On Atwood's Machine with a Nonzero Mass String

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Let us consider a classical high school exercise concerning two weights on a pulley and a string, illustrated in Fig. 1(a). A system like this is called an Atwood's machine and was invented by George Atwood in 1784 as a laboratory experiment to verify the mechanical laws of motion with constant acceleration.[1] Nowadays, Atwood's machine is used for didactic purposes to demonstrate uniformly accelerated motion with acceleration arbitrarily smaller than the gravitational acceleration g. The simplest case is with a massless and frictionless pulley and a massless string. With little effort one can include the mass of the pulley in calculations. The mass of a string has been incorporated previously in some considerations and experiments.[2-7] These include treatments focusing on friction, justifying the assumption of a massless string,[6] incorporating variations in Earth's gravitational field,[5] comparing the calculated value of g based on a simple experiment,[3] taking the mass of the string into account in such a way that the resulting acceleration is constant,[2,4] or in one exception[7] solely focusing on a heavy string, but with a slightly different approach. Here we wish to provide i) a derivation of the acceleration and position dependence on the weights' masses based purely on basic dynamical reasoning similar to the conventional version of the exercise, and ii) focus on the influence of the string's linear density, or equivalently its mass, on the outcome compared to a massless string case.

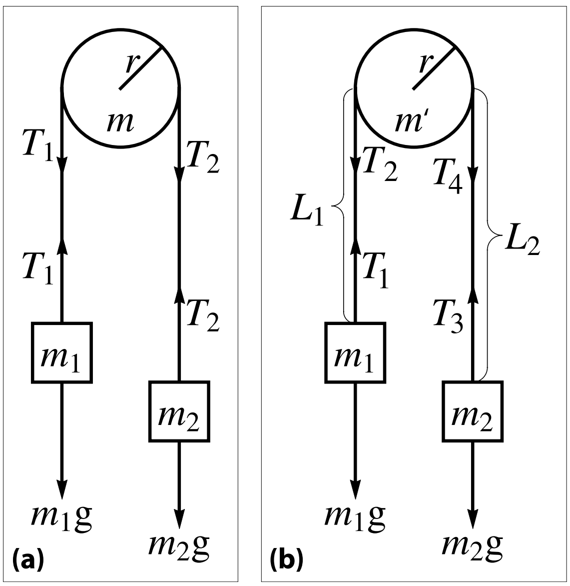

Fig. 1. (a) Tensions for a massless string and (b) for a nonzero mass string.

Conventional systems

The aim of the exercise is to calculate the acceleration of the weights. In the simplest case, when the mass of the pulley is $m = 0$ and the string is massless, it is easy to find the acceleration:

$$a = \frac{m_1 - m_2}{m_1 + m_2} g, \tag{1}$$

where we assumed, without loss of generality, that $m_1 > m_2$. In a slightly more complex situation, when the mass of the pulley is nonzero, we need to take into account its moment of inertia I. Assuming the pulley is a uniform disk $I = \frac{1}{2}mr^2$, where r is the radius of the pulley, again, it is not difficult to find the acceleration to be

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m} g. \tag{2}$$

Directing the x-axis downward and placing its origin on the level of the center of a pulley, the time-dependent position of a weight with mass m_1 can be found based on a well-known formula describing a uniformly accelerated motion:

$$x(t) = \frac{at^2}{2} + L_0, \tag{3}$$

where L_0 is the initial position. Yet how will such a system behave if the mass of the string m_0 is nonzero?

Nonzero mass string

The tension of the string applied to the point of suspension of a weight and to the point where the string stops touching the pulley are not the same, as indicated in Fig. 1(b). Tension T_1 is caused by the mass m_1, while tension T_2 is caused by the mass m_1 and the mass of a string with length L_1—similarly for the weight with mass m_2. Therefore, we have four different tensions, not two like in the case of a massless string. We begin by finding the equations of motion for the masses m_0, m_1, m_2, and m.

Let the string have a total length L. It is compounded of lengths of individual segments:

$$L = L_1 + L_2 + \pi r. \tag{4}$$

Of course L_2 depends on L_1 and vice versa:

$$L_2 = (L - \pi r) - L_1 \equiv L' - L_1. \tag{5}$$

Let us call L' a *reduced length* of a string. Let us assume also that the linear density of a string $\rho = \frac{m_0}{L}$ is constant. We then divide our system into five subsystems: two weights, two segments of string, and a pulley. According to Newton's second law of motion, the equations of motion for the weights

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and pulley are the same as in the case of a massless string:

$$m_1 g - T_1 = m_1 a, \tag{6}$$

$$T_3 - m_2 g = m_2 a, \tag{7}$$

$$r(T_2 - T_4) = I'\alpha \equiv \frac{1}{2} m' r^2 \alpha, \tag{8}$$

with a well-known relation between angular and linear accelerations,

$$a = r\alpha, \tag{9}$$

where α denotes the angular acceleration and I' is the moment of inertia of the pulley with a radius r and a segment of a string (with mass $\rho\pi r$) touching the pulley. This leads to

$$I' = I + (\rho\pi r) r^2 = \frac{1}{2} m r^2 + (\rho\pi r) r^2 = \frac{1}{2} m \left(1 + \frac{2\rho\pi r}{m}\right) r^2$$

and an *effective mass*

$$m' = m\left(1 + \frac{2\rho\pi r}{m}\right).$$

For a pulley much heavier than the string (which is usually the case), we have $m' \approx m$. Assuming $T_1 = T_2$ and $T_3 = T_4$ we get, after solving the above system of equations, the acceleration given by Eq. (2). However, in the nonzero mass string case, Newton's second law gives us two more equations: one for each segment of the string. They are as follows:

$$T_1 - T_2 + \rho L_1 g = \rho L_1 a, \tag{10}$$

and

$$T_4 - T_3 + \rho L_2 g = \rho L_2 a. \tag{11}$$

Now we can solve the system (6)–(11) to find the acceleration a. From Eqs. (6), (7), (10), and (11) we evaluate tensions, insert them into Eq. (8), and use Eq. (9) to eliminate r. Eventually, we get the following formula:

$$a = \frac{\frac{m_0}{L}(2L_1 - L') + m_1 - m_2}{\frac{m_0}{L} L' + m_1 + m_2 + \frac{1}{2} m'} g. \tag{12}$$

Again if we set $m_0 = 0$, we get Eq. (2), so our solution properly reduces to a simpler case.

Further dynamical considerations

Next, we want to find the dependence of path on time when the weights move with the above acceleration (12). The last formula was derived for arbitrary L_1, which is the x coordinate. Hence, acceleration a is *proportional* to the position; in general, acceleration is a second derivative of the position with respect to time. Thus, we get the following differential equation:

$$\ddot{x} = Ax + B, \tag{13}$$

where

$$A = \frac{\frac{2gm_0}{L}}{\frac{m_0}{L} L' + m_1 + m_2 + \frac{1}{2} m'}, \tag{14}$$

$$B = \frac{g\left(-\frac{m_0}{L} L' + m_1 - m_2\right)}{\frac{m_0}{L} L' + m_1 + m_2 + \frac{1}{2} m'}. \tag{15}$$

are the coefficients of L_1 and the free term in Eq. (12), respectively, and we put $x \equiv L_1$. A solution of this differential Eq. (13) has the following form:

$$x(t) = C_1 e^{-\sqrt{A}t} + C_2 e^{\sqrt{A}t} - \frac{B}{A}, \tag{16}$$

which can be solved in two steps.[8] First, take the corresponding homogeneous equation $\ddot{x} = Ax$ and assume a solution of a form $x \sim e^{\lambda t}$. Inserting this into the last equation, we get $\lambda^2 e^{\lambda t} - Ae^{\lambda t} = 0$ with a characteristic equation $\lambda^2 - A = 0$. Hence, $\lambda = \pm\sqrt{A}$ and the first part of the solution is

$$x_H(t) = C_1 e^{\sqrt{A}t} + C_2 e^{-\sqrt{A}t}.$$

The particular integral has to be a constant,

$$x_P(t) = const. = -\frac{B}{A},$$

which is easy to verify by insertion into Eq. (13). The solution is a sum $x(t) = x_H(t) + x_P(t)$ and takes the form of Eq. (16); C_1 and C_2 are integration constants that can be determined by introducing initial conditions. For example, let the weight with mass m_1 have a position x_0 at time $t = 0$, and let the initial velocity be equal to zero. These assumptions lead to $C_1 = C_2$ and

$$C_1 = \frac{1}{2}\left(x_0 + \frac{B}{A}\right). \tag{17}$$

After inserting $C_1 = C_2$ and Eq. (17) into Eq. (16), the solution can be written in a compact form:

$$x(t) = \left(x_0 + \frac{B}{A}\right)\cosh\left(\sqrt{A}t\right) - \frac{B}{A}. \tag{18}$$

Now, what will happen if we want to use Eq. (18) for the case of a massless string? Based on Eq. (14), this means taking $A = 0$ in the position Eq. (18), but putting simply cosh0 =1 is not sufficient, as we also have two $\frac{B}{A}$ terms, which are indeterminate for $A = 0$. The proper way is to expand Eq. (18) into a Taylor series[9] around $A = 0$ (Maclaurin series):

$$\begin{aligned} x(t) &= \left(x_0 + \frac{B}{A}\right)\cosh\left(\sqrt{A}t\right) - \frac{B}{A} \\ &\simeq \left(x_0 + \frac{B}{A}\right)\left[1 + \frac{1}{2}\left(\sqrt{A}t\right)^2\right] - \frac{B}{A} \\ &= \left(x_0 + \frac{B}{A}\right)\left[1 + \frac{1}{2}At^2\right] - \frac{B}{A} \\ &= x_0 + \frac{1}{2}x_0 A t^2 + \frac{1}{2}Bt^2 \\ &\xrightarrow[A\to 0]{} x_0 + \frac{1}{2}Bt^2 \end{aligned} \tag{19}$$

We need to remember that m_0 is also present in *B*; setting $m_0 = 0$ reduces Eq. (15) correctly to Eq. (2), which means that Eq. (19) is equivalent to Eq. (3). This means that the solution Eq. (16) is general enough to incorporate also a massless string case.

Numerical illustration

Figure 2 displays plots of two position dependencies of a weight with mass m_1: quadratic, obtained with the assumption $m_0 = 0$, and hyperbolic for the case $m_0 > 0$. The plots are for mass ratios $m' : \mathrm{m}_1 : m_2 : m_0 = 2{:}1{:}0.5{:}0.02$; these were chosen arbitrarily, but keeping them realistic enough to present a possible laboratory experiment outcome.

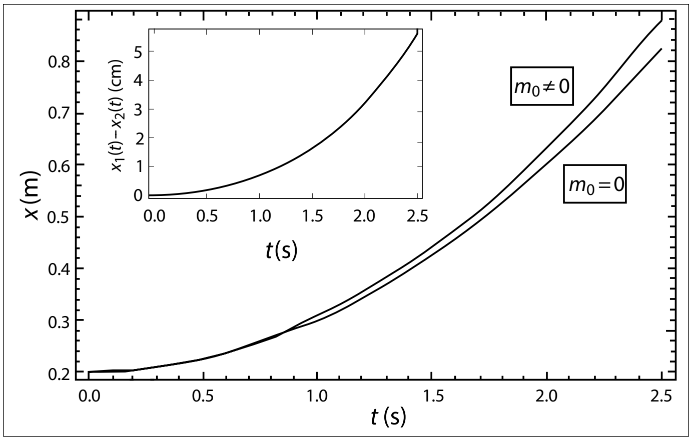

Fig. 2. Dependence of position of a weight on time for a nonzero mass [$x_1(t)$] and massless [$x_2(t)$] string. The inset shows the difference between solutions. The mass m_1 =1 kg. Note there are various units on the vertical axis.

Intuitively, a moving weight in the case of a string with mass should move faster than when the string is massless, and indeed Fig. 2 confirms this hypothesis. On the other hand, for values of the above parameters the difference is small but measurable. George Atwood in his experiment assumed that both the pulley and the string are massless, but this did not prevent him from achieving satisfactory results.

Conclusions and experimental design

A formula (12) for acceleration in the case of a nonzero mass string was derived. Next, the equation of motion (13) was formulated and solved to obtain a path-time relationship (18). The solution was compared with that obtained for a massless string (3). It was found that the mass of a string results in the weight falling *faster* than when the string is massless. Using arbitrary but reasonable values of the masses, we showed that the difference in behavior may easily be observable.

Students and teachers are encouraged to perform comparative experiments using strings with different linear densities. For example, one can use a very light string (*approximately* massless) and a light chain. Next, after precise measurement of the falling time, the initial distance may be changed in order to plot several points to form a diagram similar to Fig. 2. Plotting results for different strings will allow i) direct comparisons of the influence of the string's linear density on the motion and ii) fitting the $x(t)$ function given by Eq. (18) to verify the initially measured parameters of the system. Due to easily taking into account the mass of the pulley, one can ascribe the differences mainly to the effects coming from the string's mass (if one can justify an approximation of a frictionless axle; if not, it has been already described[6] how to take friction into account).

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9. For a detailed explanation see any introductory textbook on calculus, e.g., Morris Tenenbaum and Harry Pollard, *Ordinary Differential Equations* (Dover Publications, Inc., New York, 1985), pp. 535–536.

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