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Robust estimation of the range-based GARCH model: Forecasting volatility, value at risk and expected shortfall of cryptocurrencies

Piotr Fiszeder [a,b,*], Marta Małecka [b,c], Peter Molnár [a,b,d]

[a] Faculty of Economic Sciences and Management, Nicolaus Copernicus University in Torun, Torun, Poland
[b] Faculty of Finance and Accounting, Prague University of Economics and Business, Prague, Czech Republic
[c] Faculty of Economics and Sociology, University of Łódź, Łódź, Poland
[d] UiS Business School, University of Stavanger, Stavanger, Norway

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ABSTRACT

Traditional volatility models do not work well when volatility changes rapidly and in the presence of outliers. Therefore, two lines of improvements have been developed separately in the existing literature. Range-based models benefit from efficient volatility estimates based on low and high prices, while robust methods deal with outliers. We propose a range-based GARCH model with a bounded M-estimator, which combines these two improvements with a third new improvement: a modified robust method, which adds elasticity in treating the outliers. We apply this model to Bitcoin, Ethereum Classic, Ethereum, and Litecoin and find that it forecasts variances, value at risk, and expected shortfall more accurately than the standard GARCH model, the standard range-based GARCH model, and the GARCH model with the robust estimation. Utilization of high and low prices joined with a novel treatment of outliers makes our model perform well during extreme periods when traditional volatility models fail.

1. Introduction

The body of literature regarding cryptocurrencies is extensive. The review papers Corbet et al. (2019), Kyriazis (2019), Merediz-Solà and Bariviera (2019), Bariviera and Merediz-Solà (2021), Kayal and Rohilla (2021), Almeida and Gonçalves (2022), Fang et al. (2022) cite several hundred articles devoted to cryptocurrencies. One of the key research topics in these papers is volatility models. Dozens of studies have been dedicated to this issue. In most of them, various GARCH (generalized autoregressive conditional heteroscedasticity) models have been applied (see references in Corbet et al., 2019; Alexander and Dakos, 2020; Bariviera and Merediz-Solà, 2021; Almeida and Gonçalves, 2022; Dudek et al., 2024). The importance of this topic stems from the fact that volatility modeling is closely related to the estimation of risk. Volatility forecasts can also be applied to the construction of portfolios, hedging, and option pricing. On January 10, 2024, the U.S. Securities and Exchange Commission approved the launch of the first Bitcoin exchange-traded funds. The introduction of these new instruments increases investors' interest in investment strategies in which cryptocurrency volatility forecasts can also be used.

So far, various studies have pointed to different models, and there is no consensus among researchers regarding either the choice of the appropriate form of the model or the accuracy of the volatility forecasts. In this paper, we apply two promising approaches to forecast the volatility of cryptocurrency returns, and based on both approaches, we propose a new method. The two applied approaches are: robust estimation methods and range-based volatility models.

A well-known empirical observation about cryptocurrency dynamics is that there are periods of huge volatility. These periods are often associated with the presence of outliers. Frequently reported reasons for the occurrence of outliers include cybercrime, hacks, unsuccessful fork attempts, and regulatory disorientation (Chaim and Laurini, 2018; Corbet et al., 2019). Such events cause rapid price downswings. Moreover, cryptocurrencies exhibit explosive behaviors in multiple periods when they achieve extraordinary value gains. (Bouri et al., 2019; Hall and Jasiak, 2024). The effects of outliers on volatility modeling are

* Corresponding author. Faculty of Economic Sciences and Management, Nicolaus Copernicus University in Torun, Gagarina 13a, 87-100, Torun, Poland.
E-mail addresses: piotr.fiszeder@umk.pl (P. Fiszeder), marta.malecka@uni.lodz.pl (M. Małecka), peto.molnar@gmail.com (P. Molnár).

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wide-ranging. First, the standard GARCH models cease to fit to the time series (Charles and Darné, 2019). Second, the volatility forecasts based on the GARCH models become less accurate (Trucíos, 2019). Unfortunately, the presence of outliers was considered in only a few papers (Catania et al., 2018; Charles and Darné, 2019; Trucíos, 2019; Hung et al., 2020; Catania and Grassi, 2022). In those papers, robust methods were used to forecast volatility. Catania et al. (2018) and Catania and Grassi (2022) applied the score-driven model with the generalized hyperbolic skew Student's t-distribution. Trucíos (2019) used robust methods for the GARCH model and the generalized autoregressive score models. Hung et al. (2020) employed the realized GARCH model with jump-robust realized measures. These papers showed that the forecasts based on the robust methods were more accurate than those from the standard GARCH models.

The second promising approach to model the volatility of returns is to apply the range-based models. These models are built on the ranges or range-based volatility estimators (for a detailed review of the range-based volatility estimators, see Molnár, 2012). This means that in addition to, or instead of, daily closing prices, they use daily high and low prices. This increases the scope of applied information. At the same time, such models do not require intraday data. Instead, the data needed for estimation are commonly available for most financial assets. The forecasts based on these models are usually more accurate than the forecasts from the traditional models built on closing prices (e.g. Fiszeder and Perczak, 2016; Molnár, 2016; Xie, 2019; Fiszeder and Małecka, 2022; Wu and Wu, 2022; Fałdziński et al., 2024).

The range-based models have very rarely been used for modeling cryptocurrencies. Tan et al. (2020) proposed the asymmetric bilinear conditional autoregressive range (ABL-CARR) model for the Garman-Klass estimator (Garman and Klass, 1980) and applied it to 102 cryptocurrencies to describe their volatility. Wu et al. (2020) introduced the two-component CARR (CCARR) model and used it to forecast the volatility of Bitcoin prices. Forecasts from this model were more accurate than the ones from the return-based GARCH models.

To the best of our knowledge, the previous attempts to utilize the range-based GARCH models to forecast volatility under extreme conditions have never been used in combination with robust estimators. These two lines of the research, each improving the effectiveness of variance forecasting differently, were developed separately. Our paper fills this gap by proposing an approach that links both.

This study has three main contributions. The first is to combine the range-based volatility model with a robust estimation method and suggest a new approach to model volatility. To this aim, we correct the bounded M-estimator (BM-estimator) of Muler and Yohai (2008) and use it for the RGARCH model of Molnár (2016). This correction includes the adjustment of parameters of the robust method to the changed form of the volatility equation and to the properties of the cryptocurrencies, which contain periods of very high volatility with many outliers. Combining the BM-estimator and RGARCH model brings significant advantages over the existing methods. Through using low and high prices, it introduces additional information about variability throughout the day. However, simultaneously, it is not as sensitive to outliers as the standard range-based models estimated with the use of the maximum likelihood method. Our proposition is methodologically different from the one suggested by Ke et al. (2021), who applied the robust minimum distance estimator to the CARR model of Chou (2005).

Our second contribution is the modification of the propagation mechanism of Muler and Yohai (2008), which restricts the effect of an outlier on the estimated conditional variance. We suggest not limiting the influence of such an outlier on the predetermined value, as done in their procedure, but decreasing its influence proportionally. As a result of this modification, the forecasts of volatility on the days directly following the shock remain at a slightly increased level compared to those based on the Muler and Yohai (2008) approach. This ensures consistency with empirical observations of financial markets that after very large daily price movements, volatility remains elevated for several days.

The third contribution is to show that forecasts of volatility and risk based on the proposed methods are more accurate than forecasts from the following three benchmarks: the standard GARCH model, the standard range-based GARCH model, and the GARCH model with robust estimation. We compare the empirical volatility forecast errors based on both the proposed and the existing models for selected main cryptocurrencies. To formally evaluate the differences among the models, we apply the test of equal conditional predictive ability by Giacomini and White (2006) and the model confidence set (MCS) procedure developed by Hansen et al. (2011). To gain further insight into the sources of comparative advantages of the models, we use the decomposition of Rossi and Sekhposyan (2011). We further present the application of the volatility forecasts to calculate risk measures, value at risk (VaR), and expected shortfall (ES). To evaluate the obtained risk forecasts, we use a statistical testing procedure, which includes standard separate VaR and ES tests and the encompassing test by Dimitriadis et al. (2023), where VaR and ES forecasts are used simultaneously. To assess the utility of risk forecasts in business practice, we apply the MCS procedure based on loss functions proposed by Sarma et al. (2003) and Caporin (2008).

The remainder of the paper is structured as follows. Section 2 provides the theoretical background and introduces the proposed models and estimation methods. In Section 3.1, the data are introduced and described. Section 3.2 analyses the influence of the considered methods on the values of the model parameters. In Section 3.3, the variance forecasts are formulated and evaluated. Section 3.4 contains the robustness check for various values of parameters used in the proposed procedure. In Section 4, the forecasts of VaR and ES are presented and evaluated. Conclusions are given in Section 5.

2. Theoretical background

2.1. Existing models and estimation methods

Let us assume that $\varepsilon_t = h_t^{0.5} z_t$, where $z_1, z_2, \ldots, z_T$ are i.i.d. random variables with a continuous density f such that $E(z_t) = 0$ and $var(z_t) = 1$. The standard GARCH (p, q) model introduced by Bollerslev (1986) can be written as:

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{1}$$

where $\varepsilon_t = r_t - \gamma_0$, r_t is the return, γ_0 is the constant, h_t is the conditional variance. The usual restrictions are $\alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0$ (for $i = 1, 2, \ldots, q$; $j = 1, 2, \ldots, p$), however, non-negativity of the conditional variance can also be ensured with weaker conditions (see Nelson and Cao, 1992). A necessary and sufficient condition for strict stationarity of the process ε_t with finite variance is $\alpha_1 + \ldots + \alpha_q + \beta_1 + \ldots + \beta_p < 1$.

This standard GARCH model is based only on closing prices, which limits the efficiency of the resulting volatility estimates. More efficient estimates of variance can be received from the range-based models. One such model is the range GARCH model (RGARCH) by Molnár (2016). It can be formulated as:

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \sigma_{P\ t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{2}$$

where $\sigma_{P\ t}^2$ is the Parkinson estimator of variance (Parkinson, 1980). This estimator is given as $\sigma_{Pt}^2 = [ln(H_t/L_t)]^2/(4\ln 2)$, where H_t and L_t are the high and low prices over a day, respectively. The parameter restrictions for this model are analogous to those of the standard GARCH model.

The parameters of both the GARCH and RGARCH models are commonly estimated by the maximum likelihood (ML) method. If we assume that the distribution of z_t is normal, then the log-likelihood function can be written as:

$$L(\varsigma) = -\left(\frac{T}{2}\right)\ln(2\pi) - 0.5\sum_{t=1}^{T}\left(\ln h_t(\varsigma) + \frac{\varepsilon_t^2}{h_t(\varsigma)}\right), \tag{3}$$

where $\varsigma = (\alpha_0, \alpha_1, .., \alpha_q, \beta_1, \beta_2, ..., \beta_p)$ is a vector containing unknown parameters of the model, T is the number of daily observations used in estimation. For the Student's t-distribution of z_t, the log-likelihood function changes to:

$$L(\varsigma) = T\ln\Gamma\left(\frac{\nu+1}{2}\right) - T\ln\Gamma\left(\frac{\nu}{2}\right) - \left(\frac{T}{2}\right)\ln(\pi\nu) - 0.5\sum_{t=1}^{T}\ln h_t(\varsigma)$$
$$+ -\left(\frac{\nu+1}{2}\right)\sum_{t=1}^{T}\ln\left(1 + \frac{\varepsilon_t^2}{\nu h_t(\varsigma)}\right). \tag{4}$$

It is commonly known that extremely large observations cannot be explained by standard volatility models like the GARCH model, even when heavy-tailed conditional distributions of an error term are assumed (see, e.g., Franses and Ghijsels, 1999). More importantly, the extreme observations affect the identification and estimation of the GARCH models. Carnero et al. (2007), Catalán and Trívez (2007), Carnero et al. (2012), and Boudt et al. (2013) show that the presence of outliers in GARCH processes results in biases on the maximum likelihood parameter estimates as well as on the volatility estimates. If such outliers are not treated adequately, they can considerably deteriorate the forecasting accuracy (Catalán and Trívez, 2007; Trucíos and Hotta, 2015). A possible way to solve this problem is to apply robust estimation methods. One of the most frequently used robust estimators of the GARCH model parameters is the bounded M-estimator (BM) of Muler and Yohai (2008).

To gain robustness, Muler and Yohai (2008) modified the GARCH model by including a mechanism that restricts the propagation of the outlier effect on the estimated conditional variance:

$$h^*_{t,k} = \alpha_0 + \sum_{i=1}^{q}\alpha_i h^*_{t-i,k}\, s_k\left(\varepsilon^2_{t-i} \Big/ h^*_{t-i,k}\right) + \sum_{j=1}^{p}\beta_j h^*_{t-j,k}, \tag{5}$$

where $s_k(u) = \begin{cases} u \text{ if } u \le k, \\ k \text{ if } u > k \end{cases}$.

In this formula, s_k serves as a robust filter, in which k is a chosen fixed parameter responsible for the scale of filtering.

The robust filter in the procedure proposed by Muler and Yohai (2008, MY procedure) is used together with the mechanism that limits the impact of the outliers on the objective function. This procedure is based on the transformation of ε_t^2 and z_t^2: $y_t = \ln(\varepsilon_t^2)$ and $w_t = \ln(z_t^2)$. If the density f of z_t is symmetric around 0, then the density of w_t is given by:

$$g(w) = f\left(e^{w/2}\right) e^{w/2}. \tag{6}$$

In particular, when f is the $\mathrm{N}(0,1)$ distribution, then $g = g_0$ has the form:

$$g_0(w) = (2\pi)^{-0.5} e^{-0.5(e^w - w)}. \tag{7}$$

The value of the density of g_0 is further transformed by the ρ_0 function: $\rho_0 = -\ln(g_0)$. Finally, the values of ρ_0 are used in the objective functions M and M^*:

$$M(\varsigma) = -\left(\frac{T}{2}\right)\ln(2\pi) + \frac{1}{T\sum_{t=1}^{T}\rho(y_t - \ln h_t(\varsigma))}, \tag{8}$$

$$M^*(\varsigma) = -\left(\frac{T}{2}\right)\ln(2\pi) + \frac{1}{T}, \tag{9}$$

where $\rho = m(\rho_0)$ and m is a bounded nondecreasing function. The BM estimator is obtained by minimizing and comparing these objective functions:

$$\hat{\varsigma} = \begin{cases} \hat{\varsigma}_1 \text{ if } M(\hat{\varsigma}_1) \le M^*(\hat{\varsigma}_2), \\ \hat{\varsigma}_2 \text{ if } M(\hat{\varsigma}_1) > M^*(\hat{\varsigma}_2), \end{cases} \tag{10}$$

where $\hat{\varsigma}_1 = \arg\min_{\varsigma \in C} M(\varsigma)$, $\hat{\varsigma}_2 = \arg\min_{\varsigma \in C} M^*(\varsigma)$ for some convenient compact set C.

The m function in the above estimator is responsible for limiting the influence of outliers. This function composed with ρ_0 gives the ρ function. Muler and Yohai (2008) considered two BM estimators with the following two ρ and m functions:

$$\rho_1 = m_1(\rho_0), \tag{11}$$

where $m_1(x) = \begin{cases} x & \text{if } x \le a, \\ P(x) & \text{if } a < x \le b, \\ 4.15 & \text{if } x > b, \end{cases}$ $P(x) = \frac{2}{(b-a)^3}\Big[\frac{1}{4}(x^4 - a^4) - \left(\frac{1}{3}\right)(2a+b)(x^3 - a^3) + \left(\frac{1}{2}\right)(a^2 + 2ab)(x^2 - a^2)\Big] - \frac{2a^2 b}{(b-a)^3}(x-a) - \frac{1}{3(b-a)^{-2}}(x-a)^3 + x$ and $a = 4$ and $b = a + 0.3$,

$$\rho_2 = m_2(\rho_0), \tag{12}$$

where $m_2(\nu) = 0.8 m_1(\nu/0.8)$.

The function m_1 coincides with a fourth-order polynomial in the interval $[4, 4.3]$. It is a smoothed version of

$$m(x) = \begin{cases} x \text{ if } x \le 4, \\ 4 \text{ if } x > 4. \end{cases} \tag{13}$$

Muler and Yohai (2008) found in the Monte Carlo simulations that the BM_2 estimator (the estimator variant with the m_2 function given in equation (12)) is less efficient but more robust than the BM_1 estimator (the estimator variant with the m_1 function given in equation (11)). These two variants of the estimator, which depend on the choice between the m_1 and m_2 functions, are applied by the authors with different values of parameter k. For m_1 they take $k = 5.02$ in equation (5). This value is such that $P(z_t^2 < 5.02) = 0.975$, when $z_t = \varepsilon_t/\sqrt{h_t}$ is $\mathrm{N}(0,1)$ and, consequently, $z_t^2 = \varepsilon_t^2/h_t$ is χ_1^2 distributed. For m_2 they chose $k = 2.71$, which is such that $P(z_t^2 < 2.71) = 0.90$, under the same assumptions. The GARCH model applied with the BM_1 or BM_2 estimator of Muler and Yohai (2008) is denoted further as BM-GARCH.

2.2. New robust approach for volatility modeling

To gain robustness but simultaneously benefit from the advantages of the range-based models, we suggest the application of the Muler and Yohai (2008) BM-estimator to the RGARCH model. We denote the RGARCH model applied with the BM estimator as BM-RGARCH. It has the following form:

$$h^*_{t,k} = \alpha_0 + \sum_{i=1}^{q}\alpha_i h^*_{t-i,k}\, s_k\left(\sigma^2_{P\ t-i} \Big/ h^*_{t-i,k}\right) + \sum_{j=1}^{p}\beta_j h^*_{t-j,k}, \tag{14}$$

where $s_k(u) = \begin{cases} u \text{ if } u \le k, \\ k \text{ if } u > k \end{cases}$.

To make the application of the MY procedure feasible to the RGARCH model and to the cryptocurrency time series, we make parameter corrections. They pertain to the value of the k parameter in the robust filter s_k (see formula (14)) and to the threshold value a in the m_1 and m_2 functions (see formulas (11) and (12)). These corrections result for two reasons. Firstly, according to our calculations, the values of k and a are not proper for cryptocurrencies, which leads to big forecasting errors. It follows from the fact that theoretical assumptions underlying the derivation of these values by Muler and Yohai (2008) are not suitable for cryptocurrencies, whose returns are more volatile and have many more outliers than other financial assets. Secondly, in equation (14) we use $\sigma^2_{P\,t}/h_t$ instead of ε_t^2/h_t (compare equation (5)). In this way, we take advantage of the properties of the Parkinson estimator of volatility $\sigma^2_{P\,t}$ at time t, which is significantly more efficient than ε_t^2. Distributions of $\sigma^2_{P\,t}/h_t$ and ε_t^2/h_t are different and have different properties. For these two reasons, we set new values to k and a. These values are based on observed empirical distributions of cryptocurrencies.

The values of the k parameter 2.71 and 5.02, considered by Muler and Yohai (2008), correspond to quantiles of order 0.9 and 0.975 from the χ^2_1 distribution. These orders are valid when the distribution of standardized returns z_t is approximately normal. In the proposed BM-RGARCH model, we set k to the empirical quintile of $\sigma^2_{P\,t}/\,h_t$, where h_t is calculated from formula (2). It means that we first estimate the parameters of the RGARCH model and then calculate its conditional variance. Next, we determine k and continue to estimate the proposed model according to the procedure described in Section 2.1. For the sake of comparability, we also replace the k parameter in the standard BM-GARCH model with its empirical counterpart. In this case, we use the empirical quantile of ε_t^2/h_t, where h_t is calculated from formula (1). We implement formulas (1) and (2) in two variants of the error term distribution: normal and Student's t. We consider the range of orders 0.9, 0.95, 0.975, and 0.99, which includes the ones considered by Muler and Yohai (2008). The empirical results, after averaging across the analyzed cryptocurrencies,[1] are given in Table 1.

For the probabilities 0.975 and 0.99, the values obtained from the empirical distributions of cryptocurrencies differ significantly from those used in the original MY procedure. For example, filtering in the standard BM-GARCH models based on a quantile of order 0.975 implies the parameter k above 6, while the original parameter from the theoretical distribution is 5.02. Filtering with the order of 0.99 would require k exceeding 12, while the theoretical distribution indicates that k is equal to 6.635. The empirical quantiles obtained for the proposed BM-RGARCH model are lower than those obtained for the standard BM-GARCH model. This fact is in line with our expectations, due to the lower volatility of the distribution based on the Parkinson estimator of variance $\sigma^2_{P\,t}$ than that of the distribution based on the squared error term ε_t^2.

Next to the k parameter of the robust filter, we consider the threshold value a used in the m_1 and m_2 functions in the BM estimator (see equations (11) and (12)). This value is responsible for changing the shape of the m_1 and m_2 functions into flat lines. It prevents the objective function from reacting sharply when outliers occur in the time series. The change into a flat line goes gradually from the threshold value a. As proposed by Muler and Yohai (2008), a is set to 4. This level corresponds to the possible values of $\rho_0(y_t - \ln h_t(\varsigma))$ (compare equation (8)), which are substituted for the m_1 and m_2 functions. The values of $\rho_0(y_t - \ln h_t(\varsigma))$ that exceed 4 have limited influence on the objective functions. The level of 4 may not be adequate for cryptocurrencies, whose distributions exhibit larger volatility than other assets. To examine this, we proceed similarly as in the case of parameter k. Namely, we use the empirical cryptocurrency time series to evaluate the quantiles of $\rho_0(y_t - \ln h_t(\varsigma))$. The chosen quantiles are then set as the threshold values a. We consider the values corresponding to the quantile orders 0.975 and 0.99.[2] The results, after averaging across the analyzed cryptocurrencies,[3] are given in Table 2.

The values presented in Table 2 show that in the BM-GARCH models, the choice of the threshold value a corresponding to the quantile of the order 0.99 (which approximately follows the choice in the original MY procedure) requires a higher value than 4, close to 6. As expected, the threshold values corresponding to the same orders are lower for the proposed BM-RGARCH models than for the standard BM-GARCH models.

All considered values of parameters k and a, together with the choice between the functions, m_1 and m_2, give a large number of parameter combinations. We examined these combinations for the cryptocurrencies described in Section 3. Finally, we made a choice, considering both the empirical forecasting error in the first in-sample period and the recommendations from the study by Muler and Yohai (2008). In this way, we set k corresponding to the order 0.975, a corresponding to the order 0.99, and the loss function to m_2. The order choices that we have made are similar to those suggested in the original MY procedure.

The proposed parameter correction method in the MY procedure can be used to analyze any time series. In our cryptocurrency study, the determined parameters do not differ significantly between individual cryptocurrencies. For this reason, we adopted average values of these parameters in our research and we suggest using them directly in the MY procedure (for the RGARCH model) in analyses of the cryptocurrency market.

2.3. *Robust volatility model with the proportional reduction of outlier impact*

The proposed BM-RGARCH model combines the form of the conditional variance equation from the RGARCH model with the robust estimation of GARCH parameters. So far, we have described corrections that serve two purposes. First, they adjust the parameters of the robust method to the modified volatility equation. Second, they adjust the joint

Table 1
Values of parameter k based on empirical quantiles vs theoretical χ^2_1 quantiles.

Order	Theoretical χ^2_1 quantiles	Values of k for BM-GARCH models		Values of k for BM-RGARCH models	
		Normal distribution	Student's t-distribution	Normal distribution	Student's t-distribution
0.900	2.706	2.277	2.199	2.230	1.966
0.950	3.841	3.937	3.813	3.619	3.263
0.975	5.024	6.622	6.498	5.842	5.295
0.990	6.635	12.686	12.672	9.344	8.472

Notes: The table presents possible values of the k parameter of the robust filter of Muler and Yohai (2008). It confronts the values proposed originally (based on the theoretical χ^2_1 quantiles), and the ones based on estimates obtained from the empirical cryptocurrency time series.

Table 2
Values of parameter a based on empirical quantiles.

Order	Values of a for BM-GARCH models		Values of a for BM-RGARCH models	
	Normal distribution	Student's t-distribution	Normal distribution	Student's t-distribution
0.975	3.285	3.232	2.958	2.733
0.990	5.992	5.985	4.474	4.087

Notes: The table presents possible values of the a parameter of the m function used in the robust method of Muler and Yohai (2008). It shows the values based on estimates obtained from the empirical cryptocurrency time series.

[1] For the details of the empirical data, see Section 3.

[2] Here, we do not consider quantiles of lower orders 0.9 and 0.95, which would have stronger influence on the objective function. This is in line with the paper by Muler and Yohai (2008), where the only used value of 4 roughly corresponds to the order of 0.99.

[3] For the details of the empirical data, see Section 3.

approach to the properties of the cryptocurrency time series. In this section, we suggest to add another feature to the proposed model. This feature changes the propagation mechanism of Muler and Yohai (2008), which restricts the effect of an outlier on the estimated conditional variance. In financial markets, volatility remains elevated for several days after very large daily price movements. To reflect this tendency, we suggest not to limit the influence of outliers to the specified value (k in equation (5)), as in the MY procedure, but decrease their influence proportionally. To implement this, we propose the BM-RGARCH model with a proportional cut, which we denote as BMpc-RGARCH:

$$h^{*}_{t,k} = \alpha_0 + \sum_{i=1}^{q} \alpha_i h^{*}_{t-i,k}\, s_k\left(\sigma^2_{P\ t-i} \Big/ h^{*}_{t-i,k}\right) + \sum_{j=1}^{p} \beta_j h^{*}_{t-j,k}, \tag{15}$$

where $s_k(u) = \begin{cases} u \text{ if } u \le k, \\ \max\,(0.3u, k) \text{ if } u > k \end{cases}$.

The value 0.3 of the proportionality ratio (hereinafter referred to as the cut parameter) was chosen to minimize the empirical forecasting error in the first in-sample period. This choice was based on the cryptocurrency data described in Section 3. We check the robustness of the cut parameter in Section 3.4. For other asset classes, it may need adjustments.

To illustrate the above problem and our proposition empirically, we use the ETC/USD data from May 2021. Cryptocurrencies plunged on May 19, 2021, after the Chinese regulators announced a cryptocurrency ban for banks and payment firms. Fig. 1 shows the presence of the outlier manifesting itself in the volatility rise from 100 to 200 in the middle of May to 1882.991 on May 19. This rise is followed by two days of highly increased volatility, May 20 and May 21. The increased volatility is observed until May 25, with a little drop on May 22. As indicated by the black line, the forecasts from the standard RGARCH model react sharply to the presence of the outlier. On the days directly after the shock, the variance forecasts reach extreme levels, which makes them very overstated. The effect of the outlier seems to be long-lasting. The volatility forecasts remain at the overstated levels even after the shock expires. This negative, long-term effect of the outlier does not appear in forecasts from the proposed robust RGARCH approach, i.e. from the BM-RGARCH and BMpc-RGARCH models. The forecasts based on these models are displayed with blue and green lines, respectively. For these models, the forecasts do not adjust to the extreme level of the outlier from May 19. This has the positive effect that the volatility in the following period is not systematically overestimated. However, for the BM-RGARCH model, the forecasts do not reflect the increased volatility on the days following directly the shock. In this model, the mechanism that restricts the propagation of the outlier effect does not depend on the outlier size. All outliers are treated in the same way, even the most extreme ones, which tend to influence the volatility forecasts in subsequent periods. The effect observed in the presented figure is that the forecasts from the model on May 20 and 21 are understated. The proportionality ratio, introduced in the BMpc-RGARCH model, offers additional elasticity. It reduces the influence of the outlier proportionally to its size, allowing it to reflect its effect on the subsequent days. This seems to be adequate for empirical situations observed after large outliers in the past. At further distance from the outlier, the forecasts from the BMpc-RGARCH model coincide with those from the BM-RGARCH model. In contrast to the overestimated forecasts from the standard RGARCH model, the forecasts from both BM models quickly come back to the level that is adequate for the observed volatility.

2.4. Robust mean in the proposed BM-RGARCH and BMpc-RGARCH models

The MY procedure, utilized in the proposed BM-RGARCH and BMpc-RGARCH models, assumes that the constant in the mean equation is equal to zero, i.e., $\gamma_0 = 0$ (see equation (1)). This assumption may not hold in practice, especially for cryptocurrencies, which have long periods of a bull market. The long-term sample means suggest values distant from zero. However, the sample mean is very sensitive to outliers. Therefore, a robust estimator of the mean is needed. We apply the corrected robust reweighted mean estimator suggested by Boudt et al. (2011). Thanks to the locality of the method, it has the advantage of reducing over-detection of outliers at times of high volatility and reducing under-detection in tranquil periods. The reweighted sample mean is given as:

$$\widehat{\gamma}_0 = \frac{\sum_{t=1}^{T} r_t I_t}{\sum_{t=1}^{T} I_t}, \tag{16}$$

where $I_t = I\left[\frac{(r_t - \text{median}_t(r_t))^2}{\text{MAD}_t^2} \le l\right]$, I is the indicator function, MAD_t is the median absolute deviation expressed as $1.483\,\text{median}_i\left(\left|r_i - \text{median}_j(r_j)\right|\right)$ and l is the threshold value. Boudt et al. (2011) consider various quantiles from the χ^2_1 distribution as the candidate values for l. Boudt et al. (2013), in their application to the robust GARCH estimation, set l to the 0.95 quantile from the χ^2_1 distribution.

We adjust the reweighted sample mean in two ways in order to match the properties of the cryptocurrency time series. First, we investigate the possible values of l, and second, we adjust the constant 1.483 in the MAD function. The values of l taken from the χ^2_1 distribution are suitable under the approximate normality of the distribution of standardized returns. This assumption is strongly violated for cryptocurrencies. Thus, instead of the χ^2_1 quantiles, we use the empirical quantiles of ε_t^2/h_t presented in Table 1. We follow the order choice of 0.95 suggested by Boudt et al. (2013).

The constant 1.483 in the MAD measure is a scale factor that ensures consistency with the standard normal distribution. This is inadequate in the case of cryptocurrencies, whose distributions are far from normal. To correct for this, we adjust the constant in the MAD function with the use of the empirical cryptocurrency time series.[4] To this end, we evaluate the standardized residuals from the considered volatility models and set the constants to such levels that the values of MAD reflect the volatility of empirical distributions. The MAD constants based on the empirical distributions, after averaging across all analyzed time series, are given in Table 3. All obtained values fit in the range 2.136–2.177, regardless of the models used to evaluate the residuals. Thus, as this constant seems insensitive to the choice of the model, we decided to set it to the average value of 2.163 in all models used in the empirical part.

To ensure the locality of the outlier detection, the reweighted sample mean uses the median_t and MAD_t evaluated in the local windows. The local window around each observation r_t is $[t - K/2, t + K/2]$. This window changes at the beginning and at the end of the sample in the following way: for $t < K/2$, the interval changes to $[1, K+1]$ and for $t > T - K/2$, it takes the form $[T - K, T]$. Following Boudt et al. (2013), we set $K = 30$. We use formula (16) to estimate the γ_0 parameter for all applied robust estimators in the empirical part.

3. Empirical study of volatility forecasts

3.1. Description of data

We apply the proposed and the competing methods to four cryptocurrency pairs: BTC/USD (Bitcoin), ETC/USD (Ethereum Classic), ETH/USD (Ethereum), and LTC/USD (Litecoin). When selecting assets, we take into account two criteria. The first criterion is connected with the market microstructure issue. To avoid its impact on volatility estimates, we choose heavily traded cryptocurrencies. The second criterion is

[4] For the details of the empirical data, see Section 3.

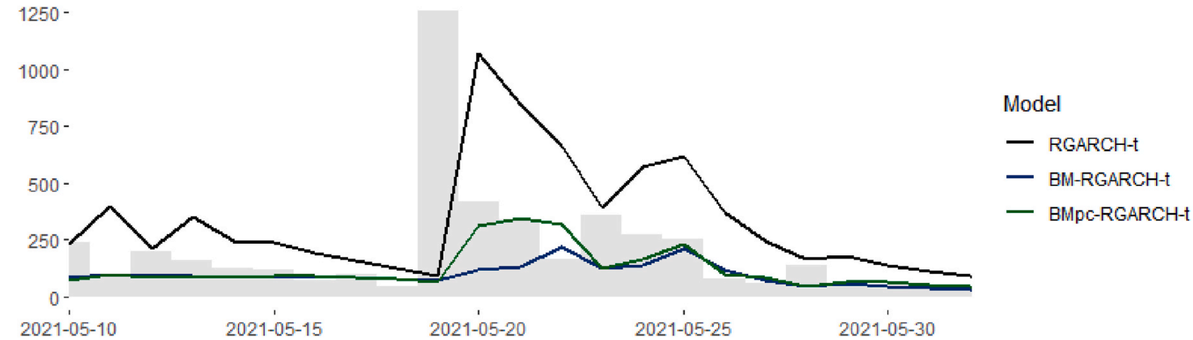

Fig. 1. Difference in the forecasting performance between the proposed robust volatility models after the presence of the outlier: ETC/USD, May 2021. Notes: The picture presents the example of the variance forecasts from the three RGARCH models displayed on the right (standard, robust and robust with the proportional cut) vs. the realized variance of the ETC/USD. Gray bars present realized variance estimated as the sum of the squared 5-min log returns. The realized variance from May 19, amounting to 1882.991, is cut at 1250 due to the size and clarity of the picture.

Table 3
MAD constants based on empirical standardized residuals vs standard normal-distribution-based constant.

Standard normal distribution	BM-GARCH model residuals		BM-RGARCH model residuals		BMpc-RGARCH model residuals	
	Normal distribution	Student's t-distribution	Normal distribution	Student's t-distribution	Normal distribution	Student's t-distribution
1.483	2.166	2.177	2.164	2.172	2.136	2.165

Notes: The table presents various values of the constant in the MAD measure used in the reweighted mean estimator of Boudt et al. (2011). These values are suited to volatility estimates obtained from the empirical cryptocurrency time series.

connected with the availability of the time series history. Its length must be sufficient for statistical evaluation of forecasts. We use daily data for building models and forecasts and intraday data to evaluate forecasts. We aim to compare volatility models and methods constructed only on daily data that are readily available. Therefore, during the estimation we only allow the use of opening, closing, low and high prices.

We analyze data received directly from the crypto exchange, Kraken. Data from coin-ranking sites can be questionable due to non-traded prices, mistakes in time stamps, use of non-fiat cross-rates, and wash trading (see Alexander and Dakos, 2020). Due to the differences in liquidity, the starting points are different for various considered cryptocurrencies. As the liquidity has been rising for all time series, we choose to set the starting points at the beginning of the first month for which the median of the number of transactions during a day is higher than 1000. According to this condition, the data start from March 1, 2016, for BTC/USD; May 1, 2017, for ETC/USD; March 1, 2017, for ETH/USD; and April 1, 2017, for LTC/USD. All series end on December 31,2021.[5]

We use log returns calculated as $r_t = 100 \ln(C_t / O_t)$, where C_t and O_t are daily closing and opening prices, respectively. Since the cryptocurrency market is open 24 h per day, there are no significant differences between open-to-close returns and close-to-close returns, and the results of our study are also valid for close-to-close returns. As a proxy of the daily variance, which we use to evaluate the forecasts, we employ the realized variance calculated as the sum of squares of the 5-min log returns. We apply the realized variance because it is a significantly more efficient estimator of the daily variance than the daily squared return or range-based variance estimators. It is a common approach to use realized variance in finance literature (see e.g., Floros et al., 2020; Reschenhofer et al., 2020; Zhang et al., 2020; Kambouroudis et al., 2021; Brini and Lenz, 2024). We apply high-frequency data only for the evaluation of forecasts but not for the estimation of models. The reason for such an approach is the desire to obtain robust methods based only on daily data that are commonly available and can be easily applied. In contrast, good quality high-frequency data are not always available, especially for illiquid cryptocurrencies. Moreover, factors related to market microstructure can significantly bias realized volatility.

Fig. 2 presents the series of prices, returns and realized variances of all analyzed cryptocurrencies. Descriptive statistics of returns and realized variances are given in Table 4.

The summary statistics show that average returns are positive and relatively high for all cryptocurrencies. It was mainly caused by the bull market in 2017, the second half of 2020, and the beginning of 2021. The volatility of Bitcoin is clearly lower compared to other cryptocurrencies. Huge outliers, both in returns and realized variances, are visible in Fig. 2. Some of the biggest outliers occurred during the COVID-19 pandemic outbreak in March 2020 and the crash after the announcement of the Chinese regulations banning banks and payment firms from using cryptocurrencies in May 2021. It justifies the application of robust methods of estimation. The distributions of almost all returns and realized variances exhibit strong skewness and high kurtosis.

3.2. Parameter estimates in volatility models

First, we verify the existence of autocorrelation in returns of analyzed cryptocurrencies. The results of the Ljung-Box test and values of the autocorrelation coefficient of order one are presented in Table 5, separately for the first estimation period (observations from 1 to 700) and the forecast period (observations from 701 till the end of the sample).[6] According to these results, autocorrelation is unstable in time and is present only for BTC/USD and ETH/USD during the first estimation period. For this reason, in the following parts, we assume the constant conditional mean and describe only the time-varying conditional volatility of returns.

Next, we check how the choice of the model influences the obtained

[5] The data is available at the following address: https://doi.org/10.18150/LH0KS7.

[6] The forecasting procedure and its results are given in Section 3.3.

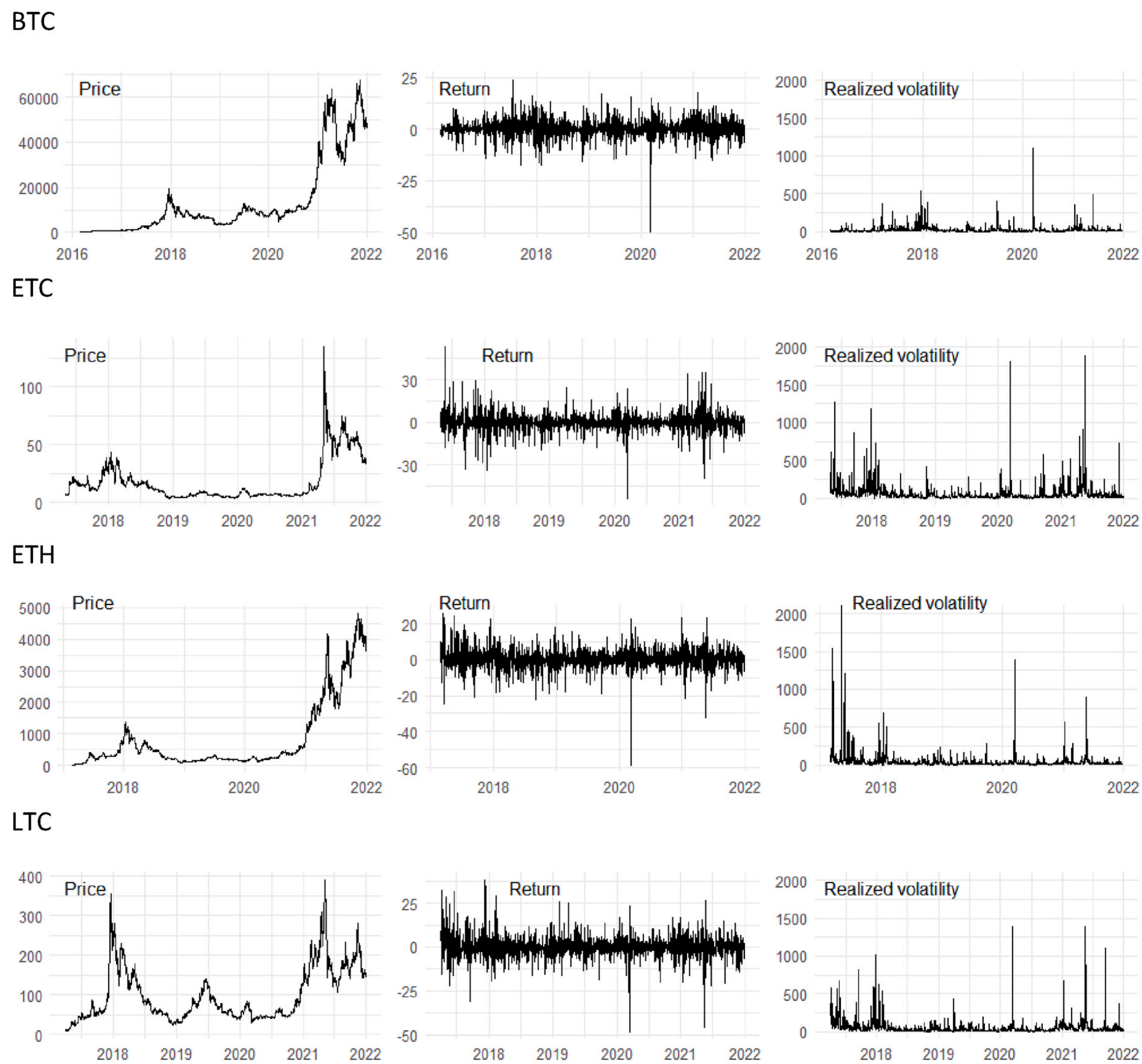

Fig. 2. Prices, returns, and realized variances of cryptocurrencies. Notes: The pictures present the prices (left panel), returns (middle panel) and realized volatilities (right panel) of the cryptocurrencies used in the empirical part of the study. The realized volatility is estimated as the sum of squared of 5-min log returns.

Table 4
Descriptive statistics of cryptocurrency daily returns and realized variances.

Cryptocurrency	Mean	Minimum	Maximum	Standard deviation	Skewness	Excess kurtosis
Returns						
BTC/USD	0.213	−49.561	23.588	4.083	−0.861	12.672
ETC/USD	0.081	−53.802	53.302	6.678	0.189	9.715
ETH/USD	0.305	−58.521	25.604	5.727	−0.541	8.691
LTC/USD	0.168	−48.035	37.801	6.197	0.043	7.455
Realized variances						
BTC/USD	20.487	0.161	1108.260	44.234	11.034	203.330
ETC/USD	62.539	0.878	1882.992	118.840	7.177	78.323
ETH/USD	48.340	0.699	16971.550	412.201	39.282	1606.947
LTC/USD	49.945	0.965	1389.869	92.012	7.323	78.124

Notes: The table presents the basic summary statistics (displayed at the top) of the returns (top panel) and realized variances (bottom panel) of the cryptocurrencies used in the empirical part of the study. The realized variance is estimated as the sum of squared 5-min log returns.

Table 5
The results of the autocorrelation test for daily standardized returns of analyzed cryptocurrencies.

Cryptocurrency	The first estimation period		The forecast period	
	LB p-value	ρ_1	LB p-value	ρ_1
BTC/USD	0.035	0.111	0.300	−0.028
ETC/USD	0.141	−0.040	0.108	0.000
ETH/USD	0.011	0.053	0.284	−0.044
LTC/USD	0.416	−0.026	0.273	−0.010

Notes: The LB p-value is the p-value of the Ljung-Box autocorrelation test with the number of lags equal to the logarithm of the number of observations. ρ_1 is the value of the autocorrelation coefficient of order one.

parameter estimates. The term volatility model, as understood here, includes the following elements: the form of the conditional mean and variance equations, the distribution of the error term, and the estimator type – whether it is the standard ML or robust BM estimator. We focus on the models that combine the RGARCH form with the BM estimator. In this case, the BM estimator is applied both in the form proposed by MY with suitable parameter corrections and in the novel form with the proportional cut. We consider two distribution variants for each model: the normal and Student's t. The latter, heavy-tailed distribution, is expected to be useful for cryptocurrencies because it gives less weight to larger innovations. As such, it can potentially mitigate the impact of extreme observations. Finally, we apply ten volatility models[7]:

(1) the standard GARCH model (equation (1)) with the normal distribution and the ML estimator (denoted as GARCH-n),
(2) the GARCH model (equation (1)) with the Student's t-distribution and the ML estimator (denoted as GARCH-t),
(3) the RGARCH model (equation (2)) with the normal distribution and the ML estimator (denoted as RGARCH-n),
(4) the RGARCH model (equation (2)) with the Student's t-distribution and the ML estimator (denoted as RGARCH-t).
(5) the GARCH model (equation (5)) with the normal distribution and the BM estimator (denoted as BM-GARCH-n),
(6) the GARCH model (equation (5)) with the Student's t-distribution and the BM estimator (denoted as BM-GARCH-t),
(7) the RGARCH model (equation (14)) with the normal distribution and the corrected BM estimator (denoted as BM-RGARCH-n),
(8) the RGARCH model (equation (14)) with the Student's t-distribution and the corrected BM estimator (denoted as BM-RGARCH-t),
(9) the RGARCH model (equation (15)) with the normal distribution and the corrected BM estimator with a proportional cut (denoted as BMpc-RGARCH-n),
(10) the RGARCH model (equation (15)) with the Student's t-distribution and the corrected BM estimator with a proportional cut (denoted as BMpc-RGARCH-t).

We estimate the parameters of ten examined models for all considered cryptocurrencies for the whole sample of data. Both the GARCH and RGARCH models are implemented with lags of order one, i.e., $p = 1$, $q = 1$ in equations ((1), (2), (5), (14) and (15). For models with the standard ML method, the parameter γ_0 is estimated jointly with other parameters (equation (3) and (4)). For models with the BM estimators, we apply the robust reweighted mean estimator given by formula (16).

Table 6
The results of parameter estimation for Ethereum Classic.

Model	γ_0	α_0	α_1	β_1
GARCH-n	0.000*	2.254*	0.128*	0.829*
GARCH-t	0.000*	2.054*	0.175*	0.825*
RGARCH-n	0.000*	6.018*	0.284*	0.552*
RGARCH-t	0.002*	3.390*	0.283*	0.717*
BM-GARCH-n	0.059	0.865*	0.103*	0.844*
BM-GARCH-t	0.064	0.987*	0.110*	0.800*
BM-RGARCH-n	0.100	1.146*	0.089*	0.810*
BM-RGARCH-t	0.124	1.722*	0.155*	0.660*
BMpc-RGARCH-n	0.100	1.280*	0.160*	0.741*
BMpc-RGARCH-t	0.124	1.143*	0.195*	0.663*

Notes: The table presents the parameter estimates obtained from various GARCH models (displayed on the left) fit to the ETC/USD time series. An asterisk * indicates significance at the 5% level. For all models with BM estimators, the γ_0 parameter is calculated based on formula (16) and treated as a fixed value when estimating other parameters, thus the significance for this parameter is not reported.

We set k corresponding to the order 0.9 (see Table 1), a corresponding to the order 0.99 (see Table 2) and the m function to m_2. We also tried the function m_1 and the forecasting results were slightly worse than these presented in the paper.[8] We present the estimation results for ETC/USD in Table 6. For the sake of brevity, we report estimated coefficients for one currency pair only (ETC/USD) to illustrate the conclusions for the sake of brevity. The conclusions regarding estimated parameters for other investigated cryptocurrencies are similar and are available from the authors upon request. Later in the paper, the main results related to forecasting are presented for all investigated cryptocurrencies.

The results demonstrate large differences in the values of parameters among the analyzed models. The values of parameter γ_0 based on the ML method, applied both to the basic GARCH and RGARCH models, are very low and give a false impression about the means. Such low values are contradictory to the summary statistics presented in Table 1. These statistics suggest that the means of returns are positive and relatively high. This is confirmed by the estimates received from the robust reweighted mean estimator applied in all models with the BM method.

The application of the BM estimator decreases the values of parameters α_0 and α_1 compared to the usage of the ML method. The decrease of parameter α_1 is weaker for the BM-pc estimator in the RGARCH model than that for the BM estimator. Parameter α_0 is responsible for the overall level of conditional volatility, while parameter α_1 for the influence of the information from the previous period on the conditional variance. The changes in these parameters may result from the reduction of the outlier influence by the BM estimator. They suggest that the presence of outliers promotes overestimation of α_0 and α_1 by the ML method, which wrongly adjusts to the extreme volatility periods (similar results were received by Ke et al., 2021, who applied the robust minimum distance estimator for the CARR model of Chou, 2005).

Another tendency observed from Table 6 is the trade-off between α_1 and β_1, which is mainly visible when comparing the GARCH and RGARCH models. It appears when comparing the standard GARCH models with the RGARCH models and occurs again when comparing the BM-GARCH models with the BMpc-RGARCH models. In these cases, the values of the parameter α_1 are much higher, and the values of the parameter β_1 are much lower for the RGARCH models compared with the GARCH models. The explanation of this effect may boil down to the fact that the Parkinson estimator of variance is less noisy than the squared return. That is why more weight is put on the new information,

[7] In all presented models with the BM estimator we apply the m_2 function given in equation (12). We consider also additional six models with the m_1 function given in equation (11) but the forecasts based on these models were slightly less accurate. The results for the models with the m_1 function are available from the authors upon request.

[8] To check the robustness of our results against the parameter choices, we have also performed comparisons for several combinations of the parameters a, k and the cut parameter. The results of this study are presented in Section 3.4.

Table 7
Evaluation of variance forecasts based on the HMSE measure.

Model	BTC/USD			ETC/USD			ETH/USD			LTC/USD		
	HMSE	MCS p-value	Rank	HMSE	MCS p-value	Rank	HMSE	MCS p-value	Rank	HMSE	MCS p-value	Rank
GARCH-n	3.233	0.000		1.561	0.000		2.261	0.000		1.847	0.000	
GARCH-t	2.500	0.000		1.743	0.000		2.468	0.000		1.772	0.000	
RGARCH-n	3.137	0.000		1.395	0.000		2.074	0.000		1.789	0.000	
RGARCH-t	2.364	0.000		1.791	0.000		2.215	0.000		1.642	0.000	
BM-GARCH-n	1.368	0.000		0.732	0.000		1.161	0.001		0.965	0.000	
BM-GARCH-t	1.294	0.000		0.704	0.000		1.048	0.021		0.865	0.011	
BM-RGARCH-n	1.136	0.000		0.647	0.082		0.905	0.172*	4	0.763	0.215*	2
BM-RGARCH-t	1.289	0.000		0.678	0.000		0.906	0.181*	3	0.937	0.006	
BMpc-RGARCH-n	**0.988**	1.000*	1	**0.632**	1.000*	1	0.852	0.977*	2	**0.744**	1.000*	1
BMpc-RGARCH-t	1.108	0.000		0.664	0.007		**0.851**	1.000*	1	0.797	0.215*	3

Notes: The table presents the forecast errors, understood as HMSE, for the considered cryptocurrencies (displayed at the top) corresponding to various GARCH models (displayed at the left). They are based on the squared difference between the forecast and the realized variance adjusted for heteroscedasticity. The realized variance is used as a proxy of variance and estimated as the sum of squares of 5-min log returns. The lowest values of HMSE are marked in bold. Each forecast error is accompanied by the p-value from the MCS test. The MCS test is performed jointly for all models. The symbol * indicates that the model belongs to the set of best models with a confidence level of 0.90. The ranking is for models belonging to the set of best models.

which translates into higher values of α_1. This observation confirms the findings reported in the literature for other financial assets (see e.g. Molnár, 2016; Fiszeder et al., 2019).

3.3. Accuracy of volatility forecasts

This section compares the forecasting performance of the ten considered models. We compute the out-of-sample one-day-ahead forecasts of variance based on each of the models.[9] The parameters of each model are estimated based on a rolling window scheme with daily parameter correction. We set the window size to 700 observations. As a proxy of the daily variance, which we use for evaluating the forecasts, we employ the realized variance calculated as the sum of squares of 5-min returns. Very similar results are also received for realized variance calculated from 15-min returns. The forecasts are evaluated based on two measures, namely the heteroscedasticity-adjusted mean squared error (HMSE) and the heteroscedasticity-adjusted mean absolute error (HMAE). These measures are chosen to take heteroskedasticity into account and, as a result, to make the forecast errors from periods of high and low volatility comparable. The correction for heteroscedasticity is especially important given the length of the examined time series and the presence of multiple periods of highly increased volatility. The HMSE loss function can be written as:

$$\mathrm{HMSE} = \frac{1}{m}\sum_{t=1}^{m}\left(1-\frac{\sigma_{F,t}^{2}}{\sigma_{R,t}^{2}}\right)^{2}, \tag{17}$$

where $\sigma_{R,t}^{2}$ is the realized variance of returns and $\sigma_{F,t}^{2}$ is the forecast of variance of returns at time t, m is the number of forecasts.

The HMAE loss function is less sensitive to outliers, which is crucial when evaluating extraordinary events. It is given as:

$$\mathrm{HMAE} = \frac{1}{m}\sum_{t=1}^{m}\left|1-\frac{\sigma_{F,t}^{2}}{\sigma_{R,t}^{2}}\right|. \tag{18}$$

The forecasting performance results are presented in Tables 7 and 8 for the HMSE and HMAE criteria, respectively. The forecast evaluation periods are: January 31, 2018 to December 31, 2021 for BTC/USD; April 1, 2019 to December 31, 2021 for ETC/USD; January 30, 2019 to December 31, 2021 for ETH/USD; and March 2, 2019 to December 31, 2021 for LTC/USD.

According to both criteria, all the forecasts based on robust methods are more accurate than the forecasts from the models with the ML estimator, regardless of whether we look at the basic GARCH or RGARCH models. The approach suggested in this paper, based on the application of corrected BM estimators to the RGARCH model, performs better than the GARCH model with the BM estimator, i.e., the BM-RGARCH-n and BMpc-RGARCH-n models are superior to BM-GARCH-n while the BM-RGARCH-t and BMpc-RGARCH-t models are usually better than BM-GARCH-t. For almost all cryptocurrencies, the lowest values of the HMSE and HMAE measures are for BMpc-RGARCH-n, which is the RGARCH model with the normal distribution combined with the corrected BM estimator with a proportional cut. The one exception is ETH for the HMSE measure, for which a slightly better result is obtained by using the BMpc-RGARCH-t model.

To formally assess the relative performance of the considered models, we apply the model confidence set (MCS) procedure developed by Hansen et al. (2011) and the test of equal conditional predictive ability by Giacomini and White (2006). The first of these procedures determines the set of best models (superior set models, SSM) with a given level of confidence. The results are given in Tables 7 and 8. According to HMSE and HMAE criteria, the only model that belongs to the SSM for all cryptocurrencies is the BMpc-RGARCH-n. For BTC/USD and ETC/USD, BMpc-RGARCH-n is the only model included in the superior set models. For LTC/USD for the HMAE measure, the set of best models includes the variants of the proposed model with and without proportional cut: BMpc-RGARCH-n and BM-RGARCH-n, and for HMSE, additionally, the variant with the proportional cut and the Student's t-distribution, BMpc-RGARCH-t. All four BM-RGARCH models (with and without proportional cut and with two variants of the error term distribution) are included in the SSM for ETH/USD. These results indicate that the forecasts of variance based on the proposed approach to model volatility are significantly more accurate than the forecasts from all considered benchmarks. In all cases, the proposed combination of the RGARCH specification with the robust estimation method gives better results than other approaches. In most cases, the variant of this procedure with the proportional reduction of outlier impact significantly dominates over the variant corresponding to the earlier methods, where outliers are limited to a fixed level.

The Giacomini and White (2006) test compares the conditional predictive ability of the four proposed models (the BM-RGARCH models with or without the proportional cut and in two distributional variants) to the relevant benchmark models. The results and the ratio of the loss functions are presented in Tables 9 and 10. If the null hypothesis of equal conditional predictive ability is rejected, then the loss ratio below one

[9] To check the robustness of our results against the out-of-sample window length, we have computed the seven-day-ahead forecasts and compared the resulting forecasting errors among the considered models. The change in the out-of-sample window length did not influence our main conclusions. For the sake of brevity, we do not include these results in the paper. They are available upon request from the authors.

Table 8
Evaluation of variance forecasts based on the HMAE measure.

Model	BTC/USD			ETC/USD			ETH/USD			LTC/USD		
	HMAE	MCS p-value	Rank	HMAE	MCS p-value	Rank	HMAE	MCS p-value	Rank	HMAE	MCS p-value	Rank
GARCH-n	1.764	0.000		0.959	0.000		1.409	0.000		1.138	0.000	
GARCH-t	1.493	0.000		1.107	0.000		1.622	0.000		1.139	0.000	
RGARCH-n	1.728	0.000		0.905	0.000		1.338	0.000		1.096	0.000	
RGARCH-t	1.509	0.000		1.212	0.000		1.489	0.000		1.074	0.000	
BM-GARCH-n	0.826	0.000		0.543	0.000		0.741	0.000		0.648	0.000	
BM-GARCH-t	0.783	0.000		0.531	0.001		0.679	0.003		0.602	0.000	
BM-RGARCH-n	0.696	0.002		0.513	0.027		0.613	0.203*	2	0.533	0.159*	2
BM-RGARCH-t	0.770	0.000		0.537	0.000		0.624	0.203*	3	0.612	0.000	
BMpc-RGARCH-n	**0.661**	1.000*	1	**0.501**	1.000*	1	**0.601**	1.000*	1	**0.524**	1.000*	1
BMpc-RGARCH-t	0.718	0.000		0.528	0.000		0.631	0.105*	4	0.564	0.007	

Notes: The table presents the forecast errors, understood as HMAE, for the considered cryptocurrencies (displayed at the top) corresponding to various GARCH models (displayed at the left). They are based on the absolute difference between the forecast and the realized variance adjusted for heteroscedasticity. The realized variance is used as a proxy of variance and estimated as the sum of squares of 5-min log returns. The lowest values of HMAE are marked in bold. Each forecast error is accompanied by the p-value from the MCS test. The MCS test is performed jointly for all models. The symbol * indicates that the model belongs to the set of best models with a confidence level of 0.90. The ranking is for models belonging to the set of best models.

Table 9
Giacomini and White test (2006) results based on the HMSE measure.

Model	BTC/USD		ETC/USD		ETH/USD		LTC/USD	
	Loss ratio	P-value	Loss ratio	P-value	Loss ratio	P-value	Loss ratio	P-value
Proposed model BM-RGARCH-n								
GARCH-n	0.124	0.000	0.172	0.000	0.160	0.000	0.469	0.000
RGARCH-n	0.131	0.000	0.216	0.000	0.191	0.000	0.486	0.000
BM-GARCH-n	0.690	0.000	0.782	0.000	0.608	0.000	0.822	0.000
Proposed model BM-RGARCH-t								
GARCH-t	0.266	0.000	0.151	0.000	0.135	0.000	0.537	0.000
RGARCH-t	0.297	0.000	0.144	0.000	0.167	0.000	0.570	0.000
BM-GARCH-t	0.993	0.461	0.928	0.069	0.747	0.001	1.017	0.720
Proposed model BMpc-RGARCH-n								
GARCH-n	0.093	0.000	0.164	0.000	0.142	0.000	0.461	0.000
RGARCH-n	0.099	0.000	0.205	0.000	0.169	0.000	0.478	0.000
BM-GARCH-n	0.522	0.000	0.744	0.000	0.539	0.000	0.808	0.000
Proposed model BMpc-RGARCH-t								
GARCH-t	0.196	0.000	0.145	0.000	0.119	0.000	0.495	0.000
RGARCH-t	0.220	0.000	0.137	0.000	0.148	0.000	0.525	0.000
BM-GARCH-t	0.733	0.001	0.889	0.020	0.660	0.000	0.937	0.003

Notes: The table presents the loss ratio between the HMAE of the proposed model (displayed at the top) and the benchmark model (displayed at the left) and the p-values of the Giacomini and White (2006) test for the considered cryptocurrencies. If the null hypothesis of equal conditional predictive ability is rejected, then the loss ratio below one implies a significant reduction of the forecast loss gained from the proposed model in relation to the benchmark one.

Table 10
Giacomini and White (2006) test results based on the HMAE measure.

Model	BTC/USD		ETC/USD		ETH/USD		LTC/USD	
	Loss ratio	P-value	Loss ratio	P-value	Loss ratio	P-value	Loss ratio	P-value
Proposed model BM-RGARCH-n								
GARCH-n	0.395	0.000	0.535	0.000	0.435	0.000	0.469	0.000
RGARCH-n	0.403	0.000	0.567	0.000	0.458	0.000	0.486	0.000
BM-GARCH-n	0.842	0.000	0.945	0.004	0.827	0.000	0.822	0.000
Proposed model BM-RGARCH-t								
GARCH-t	0.516	0.000	0.485	0.000	0.385	0.000	0.537	0.000
RGARCH-t	0.510	0.000	0.443	0.000	0.419	0.000	0.570	0.000
BM-GARCH-t	0.983	0.165	1.010	0.718	0.919	0.000	1.017	0.720
Proposed model BMpc-RGARCH-n								
GARCH-n	0.374	0.000	0.522	0.000	0.426	0.000	0.461	0.000
RGARCH-n	0.382	0.000	0.554	0.000	0.449	0.000	0.478	0.000
BM-GARCH-n	0.799	0.000	0.923	0.000	0.811	0.000	0.808	0.000
Proposed model BMpc-RGARCH-t								
GARCH-t	0.481	0.000	0.477	0.000	0.389	0.000	0.495	0.000
RGARCH-t	0.476	0.000	0.436	0.000	0.424	0.000	0.525	0.000
BM-GARCH-t	0.916	0.000	0.994	0.367	0.930	0.004	0.937	0.003

Notes: The table presents the loss ratio between the HMAE of the proposed model (displayed at the top) and the benchmark model (displayed at the left), and the p-values of the Giacomini and White (2006) test for the considered cryptocurrencies. If the null hypothesis of equal conditional predictive ability is rejected, then the loss ratio below one implies a significant reduction of the forecast loss gained from the proposed model in relation to the benchmark one.

implies a significant reduction of the forecast loss gained from the proposed model in relation to the benchmark one. The results confirm significant differences in forecasting performance for nearly all compared pairs of models. They indicate better forecasting performance of the proposed models than the benchmark ones. The only exceptions are observed for the BM-RGARCH-t models (or their variants with the proportional cut) compared to the relevant BM-GARCH-t models.

To gain further insight into the forecasting performance and the sources of advantages of the proposed models, we use the decomposition of Rossi and Sekhposyan (2011). It assigns the differences in forecasting performance to three components: time variation, predictive content, and over-fitting. The first component measures the presence of time variation in the models' performances relative to their average performances. Predictive content is based on the correlation between the in-sample and out-of-sample measures of fit. It represents the models' out-of-sample relative forecastability reflected in the in-sample relative performance. Over-fitting represents the situation when a model fits well in-sample but loses predictive ability out of sample. The results, based on the loss differences between the proposed and competing models, are presented in Tables 11 and 12. The tables give the p-values of the Rossi and Sekhposyan (2011) test conducted for the three components. They show that all the components contribute significantly to the observed better predictive ability of the proposed models. The tests for the first component reject the null of no time variation, which means there are differences between the proposed and benchmark models in their ability to forecast volatility over long periods. The tests for the second component show that the in-sample losses of the proposed models provide a better explanation of the out-of-sample losses than those of the benchmark models. The tests for the third component report significant differences in over-fitting, which means that the proportion of the out-of-sample losses that cannot be explained by the in-sample losses is different for the proposed and benchmark models.

3.4. Robustness check

In our proposed procedure, the parameters are determined from empirical quantiles. In this section, we check whether the values of the k and the a parameters significantly influence the estimation and forecasting results. For this purpose, we estimate the parameters of all models with robust estimators for various values of a and k, corresponding to the quantile orders considered in previous sections. Similarly to Section 3.2, we illustrate the estimation results for ETC/USD. These results are given in Table 13. The main conclusions presented in Section 3.2 hold for other quantile orders. First, regardless of the parameter choice, the γ_0 parameter in all BM models is clearly higher than in standard GARCH models estimated by the ML method. Second, applying the BM estimator usually decreases the values of parameters α_0 and α_1 compared to the values of these parameters obtained by the ML method. It holds for all obtained values of α_0 and almost all of α_1. Third, the changes in the α_1 and the β_1 parameters, comparing the BM-GARCH models with the BMpc-RGARCH models, are also visible.

We also check whether the forecasting performance of our models deteriorates when we change the values of the k and a parameters. The forecasting results for various quantile orders used for the calculation of the k and the a parameters are given in Tables 14 and 15, respectively, for HMSE and HMAE loss functions.

In the vast majority of cases, the lowest values of the HMSE and HMAE measures are for BMpc-RGARCH-n, which is the RGARCH model with the normal distribution combined with the corrected BM estimator with a proportional cut. In other cases, the same model with Student's t-distribution of the error term is the best. It means that the main conclusion of this study about the superiority of the suggested modeling approach remains valid for other values of quantile orders used for the calculation of the k and a parameters.

We also evaluate the forecasting performance of BMpc-RGARCH models for various values of the cut parameter. The results are presented in Tables 16 and 17, respectively, for the HMSE and HMAE criteria.

The results obtained for various values of the cut parameter in the BMpc-RGARCH models show that the forecast evaluation remains stable with changes in this parameter. Often, other values than 0.3 lead to even lower forecasting errors. This means that the forecasting conclusions are robust to the values of the cut parameter.

4. Empirical application to forecasting value at risk and expected shortfall

Accurate volatility forecasts are important because these forecasts are used to evaluate risk exposures. Two fundamental risk measures in today's finance are VaR and ES. Thus, in this section, we apply the obtained variance forecasts to produce VaR and ES predictions. To calculate VaR and ES, we use three approaches: parametric, semiparametric, and nonparametric. The risk forecasts are computed based on the same data and evaluated for the same periods as volatility forecasts: January

Table 11
Rossi and Sekhposyan (2011) test results based on the HMSE measure.

Model	Time variation in forecasting performance				Predictive content				Over-fitting			
	BTC/USD	ETC/USD	ETH/USD	LTC/USD	BTC/USD	ETC/USD	ETH/USD	LTC/USD	BTC/USD	ETC/USD	ETH/USD	LTC/USD
Proposed model BM-RGARCH-n												
GARCH-n	0.004	0.350	0.002	0.009	0.000	0.000	0.000	0.000	0.004	0.000	0.000	0.001
RGARCH-n	0.002	0.329	0.005	0.005	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000
BM-GARCH-n	0.045	0.481	0.001	0.015	0.000	0.009	0.000	0.000	0.002	0.106	0.000	0.002
Proposed model BM-RGARCH-t												
GARCH-t	0.054	0.212	0.000	0.003	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.001
RGARCH-t	0.000	0.670	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
BM-GARCH-t	0.783	0.656	0.002	0.220	0.834	0.292	0.015	0.373	1.000	0.494	0.060	0.461
Proposed model BMpc-RGARCH-n												
GARCH-n	0.004	0.381	0.002	0.014	0.000	0.000	0.000	0.000	0.003	0.000	0.000	0.001
RGARCH-n	0.002	0.341	0.006	0.006	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000
BM-GARCH-n	0.002	0.808	0.013	0.095	0.000	0.001	0.000	0.000	0.001	0.052	0.000	0.001
Proposed model BMpc-RGARCH-t												
GARCH-t	0.017	0.214	0.000	0.005	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000
RGARCH-t	0.000	0.677	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
BM-GARCH-t	0.350	0.751	0.003	0.055	0.017	0.145	0.007	0.122	0.080	0.315	0.023	0.347

Notes: The table presents the p-values of the Rossi and Sekhposyan (2011) test for the considered cryptocurrencies. The test is based on the difference in the HMSE loss function between the proposed model (displayed at the top) and the benchmark GARCH model (displayed at the left). The test is conducted for the three components, which explain potential sources of advantages of the proposed models: time variation in forecasting performance (left panel), predictive content (middle panel), and over-fitting (right panel).

Table 12
Rossi and Sekhposyan (2011) test results based on the HMAE measure.

Model	Time variation in forecasting performance				Predictive content				Over-fitting			
	BTC/USD	ETC/USD	ETH/USD	LTC/USD	BTC/USD	ETC/USD	ETH/USD	LTC/USD	BTC/USD	ETC/USD	ETH/USD	LTC/USD
Proposed model BM-RGARCH-n												
GARCH-n	0.000	0.217	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RGARCH-n	0.000	0.134	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
BM-GARCH-n	0.005	0.594	0.000	0.002	0.000	0.045	0.000	0.000	0.001	0.141	0.002	0.003
Proposed model BM-RGARCH-t												
GARCH-t	0.013	0.474	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RGARCH-t	0.000	0.501	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
BM-GARCH-t	0.190	0.128	0.001	0.005	0.511	0.616	0.037	0.713	0.805	0.771	0.244	0.832
Proposed model BMpc-RGARCH-n												
GARCH-n	0.000	0.255	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RGARCH-n	0.000	0.174	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
BM-GARCH-n	0.000	0.453	0.029	0.037	0.000	0.003	0.000	0.000	0.001	0.030	0.001	0.001
Proposed model BMpc-RGARCH-t												
GARCH-t	0.002	0.519	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RGARCH-t	0.000	0.502	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
BM-GARCH-t	0.393	0.253	0.000	0.039	0.007	0.680	0.112	0.072	0.120	0.888	0.255	0.291

Notes: The table presents the p-values of the Rossi and Sekhposyan (2011) test for the considered cryptocurrencies. The test is based on the difference in the HMSE loss function between the proposed model (displayed at the top) and the benchmark GARCH model (displayed at the left). The test is conducted for the three components, which explain potential sources of advantages of the proposed models: time variation in forecasting performance (left panel), predictive content (middle panel), and over-fitting (right panel).

Table 13
The results of the parameter estimation for Ethereum Classic for various quantile orders used for the calculation of k and a parameters.

Model	Order of a							
	0.975				0.99			
	Order of k				Order of k			
	0.9	0.95	0.975	0.99	0.9	0.95	0.975	0.99
γ_0								
BM-GARCH-n	0.059	0.080	0.142	0.071	0.059	0.080	0.142	0.071
BM-GARCH-t	0.064	0.074	0.142	0.071	0.064	0.074	0.142	0.071
BM-RGARCH-n	0.100	0.072	0.141	0.093	0.100	0.072	0.141	0.093
BM-RGARCH-t	0.124	0.063	0.113	0.107	0.124	0.063	0.113	0.107
BMpc-RGARCH-n	0.100	0.072	0.141	0.093	0.100	0.072	0.141	0.093
BMpc-RGARCH-t	0.124	0.063	0.113	0.107	0.124	0.063	0.113	0.107
α_0								
BM-GARCH-n	1.127	1.071	1.110	0.823	0.865	1.152	1.120	1.156
BM-GARCH-t	0.601	0.621	0.612	0.624	0.987	1.085	0.997	1.064
BM-RGARCH-n	0.300	0.296	0.302	0.298	1.146	0.611	0.580	0.622
BM-RGARCH-t	2.184	2.188	2.203	2.213	1.722	0.616	0.449	0.447
BMpc-RGARCH-n	1.439	0.912	0.455	0.436	1.280	1.484	2.188	0.876
BMpc-RGARCH-t	1.460	1.183	2.284	1.616	1.143	0.894	0.702	0.508
α_1								
BM-GARCH-n	0.101	0.097	0.102	0.087	0.103	0.122	0.120	0.122
BM-GARCH-t	0.101	0.102	0.099	0.102	0.110	0.127	0.118	0.126
BM-RGARCH-n	0.077	0.076	0.077	0.076	0.089	0.058	0.056	0.058
BM-RGARCH-t	0.166	0.166	0.164	0.164	0.155	0.091	0.062	0.063
BMpc-RGARCH-n	0.128	0.183	0.128	0.100	0.160	0.179	0.225	0.094
BMpc-RGARCH-t	0.464	0.302	0.231	0.100	0.195	0.230	0.170	0.113
β_1								
BM-GARCH-n	0.795	0.805	0.794	0.832	0.844	0.800	0.803	0.800
BM-GARCH-t	0.814	0.813	0.816	0.813	0.800	0.772	0.789	0.775
BM-RGARCH-n	0.823	0.825	0.824	0.824	0.810	0.868	0.874	0.867
BM-RGARCH-t	0.595	0.595	0.597	0.596	0.660	0.797	0.860	0.859
BMpc-RGARCH-n	0.647	0.663	0.751	0.782	0.741	0.692	0.587	0.805
BMpc-RGARCH-t	0.427	0.538	0.543	0.791	0.663	0.650	0.707	0.786

Notes: The table presents the parameter estimates obtained from various GARCH models with the BM estimators (displayed at the left) fit to the ETC/USD time series. An asterisk * indicates significance at the 5% level. The γ_0 parameter is calculated based on formula (16) and treated as a fixed value when estimating other parameters. Thus, the significance for this parameter is not reported.

31, 2018 to December 31, 2021 for BTC/USD; April 1, 2019 to December 31, 2021 for ETC/USD; January 30, 2019 to December 31, 2021 for ETH/USD; and March 2, 2019 to December 31, 2021 for LTC/USD.

We compute the out-of-sample one-day-ahead 1% VaR and 1% ES forecasts based on the rolling window procedure. The following equations define VaR and ES:

$$P\left(r_t < VaR_{\alpha}^{t}(r_t)\right) = \alpha, \tag{19}$$

$$ES_{\alpha}^{t}(r_t) = E\left(r_t \middle| r_t < VaR_{\alpha}^{t}(r_t)\right), \tag{20}$$

where α is the VaR level. These formulas show that VaR and ES are the quantile and the conditional expectation of the return distribution (on

Table 14
Evaluation of variance forecasts based on the HMSE measures for various quantile orders used for the calculation of k and a parameters.

Model	Order of a							
	0.975				0.99			
	Order of k				Order of k			
	0.9	0.95	0.975	0.99	0.9	0.95	0.975	0.99
BTC								
BM-GARCH-n	0.993	1.023	0.997	0.998	1.368	1.366	1.363	1.361
BM-GARCH-t	1.317	1.313	1.287	1.287	1.294	1.272	1.277	1.282
BM-RGARCH-n	0.951	0.951	0.951	0.950	1.136	1.129	1.134	1.136
BM-RGARCH-t	1.435	1.461	1.422	1.418	1.289	1.296	1.297	1.296
BMpc-RGARCH-n	**0.904**	**0.813**	**0.763**	**0.741**	**0.988**	**0.972**	**0.939**	**0.953**
BMpc-RGARCH-t	1.398	1.261	1.148	1.208	1.108	1.053	1.045	1.050
ETC								
BM-GARCH-n	0.656	0.653	0.655	0.656	0.732	0.725	0.723	0.726
BM-GARCH-t	0.703	0.730	0.739	0.747	0.704	0.702	0.706	0.703
BM-RGARCH-n	0.633	0.629	0.647	0.633	0.647	0.647	0.647	0.651
BM-RGARCH-t	0.739	0.763	0.784	0.773	0.678	0.679	0.694	0.683
BMpc-RGARCH-n	**0.623**	**0.612**	**0.598**	**0.614**	**0.632**	**0.636**	**0.630**	0.633
BMpc-RGARCH-t	0.729	0.696	0.743	0.733	0.664	0.640	0.635	**0.628**
ETH								
BM-GARCH-n	0.779	0.793	0.798	0.787	1.161	1.181	1.168	1.169
BM-GARCH-t	0.895	0.894	0.863	0.848	1.048	1.064	1.076	1.067
BM-RGARCH-n	0.681	0.711	0.686	0.672	0.905	0.879	0.916	0.933
BM-RGARCH-t	1.048	1.048	1.080	1.060	0.906	0.933	0.929	0.939
BMpc-RGARCH-n	**0.645**	**0.635**	**0.602**	**0.620**	0.852	0.858	0.844	0.894
BMpc-RGARCH-t	0.878	0.951	0.889	0.899	**0.851**	**0.853**	**0.807**	**0.822**
LTC								
BM-GARCH-n	0.740	0.721	0.720	0.719	0.965	0.937	0.931	0.930
BM-GARCH-t	0.845	0.861	0.856	0.855	0.865	0.836	0.836	0.832
BM-RGARCH-n	0.688	0.704	0.686	0.729	0.763	0.751	0.742	0.747
BM-RGARCH-t	1.004	1.019	1.039	1.032	0.937	0.926	0.888	0.891
BMpc-RGARCH-n	**0.615**	**0.605**	**0.592**	**0.624**	**0.744**	**0.671**	**0.692**	**0.699**
BMpc-RGARCH-t	0.891	0.905	0.888	0.898	0.797	0.756	0.708	0.826

Notes: The table presents the forecast errors, understood as HMSE, for the considered cryptocurrencies (displayed at the top) corresponding to various GARCH models (displayed at the left). They are based on the squared difference between the forecast and the realized variance adjusted for heteroscedasticity. The realized variance is used as a proxy of variance and estimated as the sum of squares of 5-min log returns. The lowest values of HMSE are marked in bold.

condition that the return is below VaR), respectively. Their forecasts are obtained from the formulas:

$$\widehat{VaR}_{\alpha}^{t}(\mathrm{r}_{t+1}) = \widehat{\gamma}_0 + \widehat{h}_{t+1} \cdot \widehat{VaR}_{\alpha}^{t}(z_{t+1}), \tag{21}$$

$$\widehat{ES}_{\alpha}^{t}(\mathrm{r}_{t+1}) = \widehat{\gamma}_0 + \widehat{h}_{t+1} \cdot \widehat{ES}_{\alpha}^{t}(z_{t+1}). \tag{22}$$

The mean $\widehat{\gamma}_0$ and the variance $\widehat{h}_{t+1}$ predictions in formulas (21) and (22) come from all considered models. To calculate the predictions of the quantile $\widehat{VaR}_{\alpha}^{t}(z_{t+1})$ and the conditional expectation $\widehat{ES}_{\alpha}^{t}(z_{t+1})$, we apply two approaches. In the first approach, these quantities are determined by the adopted conditional theoretical distribution (normal or Student's t). In the second approach, called filtered historical simulation, the predictions $\widehat{VaR}_{\alpha}^{t}(z_{t+1})$ and $\widehat{ES}_{\alpha}^{t}(z_{t+1})$ are estimated by the historical simulation method for standardized z_{t+1} variables. The second approach is semiparametric because it combines the parametric estimation of mean and variance and the nonparametric estimation of quantile and conditional expectation. Both approaches are commonly used in the literature (see e.g. Ghorbel and Trabelsi, 2014; Louzis et al., 2014; Zhou, 2014; Nolde and Ziegel, 2017; Shen, 2018; Patton et al., 2019; Taylor, 2019; Francq and Zakoïan, 2020; Bei et al., 2023; Müller et al., 2023). To obtain additional benchmark for evaluating the proposed models, we also compute VaR and ES forecasts by the simple historical simulation (HS) method, which represents the fully nonparametric approach. In the HS method, we directly refer to formulas (19) and (20).

To evaluate the obtained risk forecasts, we first use a statistical testing procedure and then, to assess the utility of these forecasts in business practice, we apply loss functions and compute the Basel capital charges. The statistical testing procedure includes standard separate VaR and ES tests as well as the encompassing test, where VaR and ES forecasts are used simultaneously. The initial, separate VaR testing is done to check two basic criteria. The first is the unconditional coverage criterion, which verifies whether the overall number of VaR exceedances matches the assumed VaR level. In line with the mainstream approach, we check this criterion using the Kupiec test (Kupiec, 1995). The other VaR evaluation criterion is conditional coverage, which additionally verifies the independence among VaR residuals. We check this criterion by two statistical tests that utilize two different approaches. The first test is the generalized Markov test (Pajhede, 2017), which operates on checking the Markov property in the VaR exceedance process. It is a generalization of the popular Christoffersen's Markov test (Christoffersen, 1998).

The second conditional coverage VaR test is the corrected geometric test (Malecka, 2022), which works on durations among VaR exceedances and the properties of the geometric distribution. This test utilizes the corrected asymptotic distribution applied to the geometric test statistic proposed to test VaR by Berkowitz et al. (2011). The evaluation of ES forecasts is done with two tests. The first of them utilizes the popular residual-based statistic developed by McNeil and Frey (2000), which we denote U. The second ES test, denoted U^{ES}, uses the ES-prediction-corrected variables instead of the residuals (Małecka, 2020). In both ES tests, we use one-sided p-values. Finally, we use the encompassing test by Dimitriadis et al. (2023), to simultaneously test VaR and ES forecasts from the proposed model against those from the standard models.

The evaluation procedure described above is conducted for the risk forecasts coming from all approaches: parametric, semiparametric filtered historical simulation and nonparametric historical simulation. Here, we present the results from the semiparametric method, which performs best according to the statistical tests. The results from the fully

Table 15
Evaluation of variance forecasts based on the HMAE measures for various quantile orders used for the calculation of the k and a parameters.

Model	Order of a							
	0.975				0.99			
	Order of k				Order of k			
	0.9	0.95	0.975	0.99	0.9	0.95	0.975	0.99
BTC								
BM-GARCH-n	0.628	0.640	0.631	0.631	0.826	0.825	0.825	0.825
BM-GARCH-t	0.770	0.782	0.767	0.762	0.783	0.773	0.775	0.778
BM-RGARCH-n	0.608	0.610	0.611	0.609	0.696	0.691	0.696	0.697
BM-RGARCH-t	0.873	0.894	0.862	0.860	0.770	0.775	0.775	0.775
BMpc-RGARCH-n	**0.604**	**0.562**	**0.542**	**0.527**	**0.661**	**0.642**	**0.620**	**0.616**
BMpc-RGARCH-t	0.839	0.820	0.738	0.757	0.718	0.699	0.679	0.666
ETC								
BM-GARCH-n	0.544	0.540	0.539	0.538	0.543	0.538	0.537	0.541
BM-GARCH-t	0.550	0.554	0.560	0.567	0.531	0.532	0.533	0.532
BM-RGARCH-n	0.542	0.538	0.540	0.536	0.513	0.512	0.515	0.515
BM-RGARCH-t	0.579	0.582	0.597	0.589	0.537	0.537	0.545	0.542
BMpc-RGARCH-n	**0.528**	**0.519**	**0.509**	**0.525**	**0.501**	**0.506**	0.506	0.507
BMpc-RGARCH-t	0.574	0.555	0.582	0.576	0.528	0.510	**0.505**	**0.504**
ETH								
BM-GARCH-n	0.536	0.544	0.546	0.539	0.741	0.751	0.744	0.742
BM-GARCH-t	0.580	0.592	0.577	0.572	0.679	0.686	0.689	0.685
BM-RGARCH-n	0.513	0.531	0.517	0.507	0.613	0.596	0.617	0.624
BM-RGARCH-t	0.700	0.712	0.723	0.708	0.624	0.652	0.637	0.642
BMpc-RGARCH-n	**0.503**	**0.497**	**0.476**	**0.478**	**0.601**	**0.594**	**0.581**	0.598
BMpc-RGARCH-t	0.625	0.667	0.633	0.624	0.631	0.632	0.592	**0.589**
LTC								
BM-GARCH-n	0.534	0.531	0.530	0.530	0.648	0.637	0.634	0.634
BM-GARCH-t	0.583	0.593	0.589	0.590	0.602	0.592	0.593	0.591
BM-RGARCH-n	0.515	0.520	0.512	0.525	0.533	0.526	0.522	0.524
BM-RGARCH-t	0.632	0.641	0.644	0.640	0.612	0.609	0.584	0.586
BMpc-RGARCH-n	**0.484**	**0.480**	**0.475**	**0.488**	**0.524**	**0.503**	**0.502**	**0.512**
BMpc-RGARCH-t	0.599	0.607	0.597	0.599	0.564	0.546	0.522	0.564

Notes: The table presents the forecast errors, understood as HMAE, for the considered cryptocurrencies (displayed at the top) corresponding to various GARCH models (displayed at the left). They are based on the absolute difference between the forecast and the realized variance adjusted for heteroscedasticity. The realized variance is used as a proxy of variance and estimated as the sum of squares of 5-min log returns. The lowest values of HMAE are marked in bold.

Table 16
Evaluation of variance forecasts based on the HMSE measure for the BM-RGARCH model with a proportional cut.

Model	Cut parameter				
	0.1	0.2	0.3	0.4	0.5
BTC					
BMpc-RGARCH-n	0.901	0.914	0.988	0.950	1.010
BMpc-RGARCH-t	1.001	1.011	1.108	1.074	1.142
ETC					
BMpc-RGARCH-n	0.638	0.636	0.632	0.628	0.632
BMpc-RGARCH-t	0.636	0.633	0.664	0.637	0.637
ETH					
BMpc-RGARCH-n	0.848	0.844	0.852	0.846	0.873
BMpc-RGARCH-t	0.807	0.801	0.851	0.820	0.829
LTC					
BMpc-RGARCH-n	0.679	0.681	0.744	0.687	0.696
BMpc-RGARCH-t	0.723	0.703	0.797	0.722	0.745

Notes: The table presents the forecast errors, understood as HMSE, for the considered cryptocurrencies (displayed at the top) corresponding to two BMpc-RGARCH models (displayed at the left). They are based on the squared difference between the forecast and the realized variance adjusted for heteroscedasticity. The realized variance is used as a proxy of variance and estimated as the sum of squared 5-min log returns.

Table 17
Evaluation of variance forecasts based on the HMAE measure for the BM-RGARCH model with a proportional cut.

Model	Cut parameter				
	0.1	0.2	0.3	0.4	0.5
BTC					
BMpc-RGARCH-n	0.607	0.609	0.661	0.625	0.648
BMpc-RGARCH-t	0.662	0.666	0.718	0.693	0.720
ETC					
BMpc-RGARCH-n	0.508	0.509	0.501	0.505	0.511
BMpc-RGARCH-t	0.504	0.502	0.528	0.506	0.507
ETH					
BMpc-RGARCH-n	0.585	0.581	0.601	0.583	0.593
BMpc-RGARCH-t	0.589	0.588	0.631	0.600	0.602
LTC					
BMpc-RGARCH-n	0.497	0.498	0.524	0.499	0.500
BMpc-RGARCH-t	0.529	0.521	0.564	0.525	0.530

Notes: The table presents the forecast errors, understood as HMAE, for the considered cryptocurrencies (displayed at the top) corresponding to two BMpc-RGARCH models (displayed at the left). They are based on the absolute difference between the forecast and the realized variance adjusted for heteroscedasticity. The realized variance is used as a proxy of variance and estimated as the sum of squared 5-min log returns.

parametric and fully nonparametric approaches are reported in Appendix A. The outcomes of separate VaR and ES tests are presented in Tables 18 and 19. The p-values of the VaR tests show that the VaR predictions obtained from all models considered in the study satisfy both evaluation criteria. It means that there is no model with any evidence that the overall ratio of VaR exceedances is different than the assumed VaR level or the VaR residuals are dependent on time. Unlike for VaR, the statistical tests are not always passed for ES. For instance, in the LTC/USD case, the ES forecasts are rejected for most models. The only forecasts not rejected for this cryptocurrency come from the newly proposed BM-RGARCH models with the normal or Student's t-distribution. This gives an example of a practical advantage of the proposed robust estimation method combined with the RGARCH approach. For other cryptocurrencies, the ES forecasts are usually not rejected at the 0.05 significance level (with one exception, ES forecasts from the RGARCH model with the normal distribution for ETC/USD).

Table 18
Evaluation of VaR forecasts based on the Kupiec (1995), generalized Markov (Christoffersen, 1998), and corrected geometric (Malecka, 2022) tests.

Model	BTC/USD			ETC/USD			ETH/USD			LTC/USD		
	Kupiec	Gen. Markov	Geometric	Kupiec	Gen. Markov	Geometric	Kupiec	Gen. Markov	Geometric	Kupiec	Gen. Markov	Geometric
GARCH-n	0.659	0.288	0.497	0.769	0.617	0.300	0.835	0.476	0.500	0.428	0.829	0.416
GARCH-t	0.856	0.227	0.497	0.164	0.545	0.495	0.390	0.290	0.500	0.617	0.713	0.394
RGARCH-n	0.488	0.358	0.496	0.732	0.403	0.213	0.835	0.476	0.500	0.843	0.599	0.500
RGARCH-t	0.659	0.288	0.495	0.305	0.228	0.146	0.835	0.476	0.500	0.910	0.491	0.318
BM-GARCH-n	0.488	0.358	0.496	0.305	0.479	0.500	0.920	0.582	0.498	0.843	0.599	0.500
BM-GARCH-t	0.488	0.358	0.496	0.305	0.479	0.500	0.488	0.807	0.497	0.428	0.829	0.500
BM-RGARCH-n	0.659	0.288	0.280	0.769	0.617	0.453	0.328	0.923	0.497	0.174	0.938	0.500
BM-RGARCH-t	0.856	0.227	0.450	0.551	0.733	0.445	0.920	0.582	0.499	0.139	0.153	0.500
BMpc-RGARCH-n	0.659	0.288	0.299	0.551	0.733	0.316	0.209	0.347	0.500	0.428	0.829	0.498
BMpc-RGARCH-t	0.856	0.227	0.450	0.769	0.617	0.444	0.688	0.144	0.386	0.174	0.938	0.460

Notes: The table presents the p-values of the three VaR tests (displayed at the top) for the considered cryptocurrencies (displayed at the top), corresponding to various GARCH models (displayed at the left).

Table 19
Evaluation of ES forecasts based on the residual-based test U (McNeil and Frey, 2000) and ES-prediction-corrected variables test U^{ES} (Małecka, 2020).

Model	BTC/USD		ETC/USD		ETH/USD		LTC/USD	
	U	U^{ES}	U	U^{ES}	U	U^{ES}	U	U^{ES}
GARCH-n	0.062	0.069	0.222	0.233	0.168	0.152	0.019	0.020
GARCH-t	0.069	0.065	0.127	0.131	0.133	0.147	0.020	0.029
RGARCH-n	0.115	0.101	0.030	0.021	0.096	0.094	0.010	0.019
RGARCH-t	0.069	0.058	0.367	0.350	0.122	0.113	0.022	0.036
BM-GARCH-n	0.115	0.123	0.129	0.146	0.147	0.150	0.026	0.019
BM-GARCH-t	0.075	0.072	0.059	0.063	0.468	0.486	0.037	0.045
BM-RGARCH-n	0.160	0.153	0.670	0.675	0.167	0.174	0.067	0.064
BM-RGARCH-t	0.088	0.117	0.549	0.549	0.557	0.602	0.169	0.150
BMpc-RGARCH-n	0.142	0.149	0.713	0.744	0.428	0.450	0.035	0.032
BMpc-RGARCH-t	0.075	0.057	0.500	0.501	0.662	0.663	0.025	0.021

Notes: The table presents the p-values of the two ES tests (displayed at the top) for the considered cryptocurrencies (displayed at the top), corresponding to various GARCH models (displayed at the left).

Table 20
Evaluation proposed VaR and ES models vs standard models based on the encompassing test (Dimitriadis et al., 2023).

Proposed model	Joint VaR and ES test				Auxiliary ES test				Avg. weights
	E'ing	E'd	Comb	Incon	E'ing	E'd	Comb	Incon	
Alternative model GARCH-n									
BM-RGARCH-n	2	0	2	0	1	1	1	1	(0.680, 0.514)
BMpc-RGARCH-n	1	0	3	0	2	0	2	0	(0.780, 0.756)
Alternative model GARCH-t									
BM-RGARCH-t	1	0	3	0	3	1	0	0	(0.674, 0.999)
BMpc-RGARCH-t	2	0	2	0	3	1	0	0	(0.725, 0.985)
Alternative model RGARCH-n									
BM-RGARCH-n	2	0	2	0	3	1	0	0	(0.794, 0.724)
BMpc-RGARCH-n	1	0	3	0	3	1	0	0	(0.801, 0.666)
Alternative model RGARCH-t									
BM-RGARCH-t	0	0	4	0	3	1	0	0	(0.643, 0.819)
BMpc-RGARCH-t	0	0	4	0	3	1	0	0	(0.691, 0.558)
Alternative model BM-GARCH-n									
BM-RGARCH-n	3	0	1	0	1	1	0	2	(0.705, 0.750)
BMpc-RGARCH-n	2	0	2	0	2	1	0	1	(0.728, 1.000)
Alternative model BM-GARCH-t									
BM-RGARCH-t	1	1	2	0	1	1	0	2	(0.500, 0.895)
BMpc-RGARCH-t	2	0	2	0	2	1	1	0	(0.561, 0.550)

Notes: The table summarizes the test results of the joint VaR and ES and the auxiliary ES encompassing tests based on the no-crossing link function for the considered instruments BTC/USD, ETC/USD, ETH/USD, and LTC/USD. The "E'ing" column gives the number of occurrences (out of four instruments where the proposed model (displayed at the left) encompasses a competing model (displayed at the top of the panel). The "E'd" column counts the occurrences where the proposed model is encompassed, "Comb" - neither model encompasses its competitor, and "Incon" - both models encompass each other. The column "Avg. weights" shows the estimated combination weights of the no-crossing link function ($\theta_1.\theta_2$) averaged over the four instruments.

The encompassing test of Dimitriadis et al. (2023) allows for joint evaluation of VaR and ES forecasts coming from the proposed models against those coming from alternative models. It checks whether a forecast consisting of a pair of VaR and ES performs not worse than any parametric forecast combination, including forecasts from the alternative model. The forecast combinations are given by the so-called link function. As advocated in Dimitriadis et al. (2023), we use the no-crossing link function, which prevents VaR and ES forecast crossing. The test is conducted in two variants: a joint VaR and ES test and an auxiliary ES test. In the first case, the VaR and ES forecasts are treated on an equal basis. In the second case, the test compares ES forecasts from one model to those from the alternative model, and the VaR forecasts are treated as auxiliary variables. If the test shows that the competing forecast contains no additional useful information, the tested model is referred to as encompassing. The outcomes, presented in Table 20, give the number of cases when the four proposed models encompass the alternative ones (denoted "E'ing"), the number of cases when they are encompassed by any of the alternative ones (denoted "E'd"), the number of cases where none of the models encompasses its competitor, but the combination performs best (dented "Comb") and the number of inconclusive results (denoted "Incon").

The results of the joint VaR and ES test in Table 20 show that, out of four possible outcomes, mainly two appear (with only one exception), namely either the proposed models encompass their competitors or the combination of the proposed and some alternative model performs best. It means that the proposed models contain significant information that serves to forecast VaR and ES and should be used either alone or combined with other models. The results of the ES test, when the VaR forecasts are treated only as auxiliary variables, confirm the advantages of the proposed models. In this case, though all outcomes appear in individual cases, one result is clearly dominant: the result that the proposed models encompass their competitors. It means that, in most cases, the proposed models contain all relevant information to forecast ES.

The utility of the risk forecasts in business practice is evaluated by the loss functions. We have chosen the firm's loss function FLF proposed by Sarma et al. (2003), and three firm's loss functions, C1, C2, and C3, proposed by Caporin (2008). They are, respectively, given by the following formulas:

$$l_t^{FLF} = \begin{cases} \left(r_t - \widehat{VaR}_\alpha^t(r_t)\right)^2, \text{if } r_t < \widehat{VaR}_\alpha^t(r_t), \\ -\widehat{VaR}_\alpha^t(r_t), \textit{otherwise}, \end{cases} \tag{23}$$

$$l_t^{C1} = \left|1 - \left|\frac{r_t}{\widehat{VaR}_\alpha^t(r_t)}\right|\right|, \tag{24}$$

$$l_t^{C2} = \frac{\left(|r_t| - \left|\widehat{VaR}_\alpha^t(r_t)\right|\right)^2}{\left|\widehat{VaR}_\alpha^t(r_t)\right|}, \tag{25}$$

$$l_t^{C3} = \left|r_t - \widehat{VaR}_\alpha^t(r_t)\right|. \tag{26}$$

These loss functions evaluate both the VaR level and the amount of loss when this level is exceeded. Moreover, they penalize both VaR exceedances (which correspond to risk underestimation) and excessive VaR predictions (which correspond to risk overestimation). We deem such functions relevant from the viewpoint of business practice because they consider two aspects: the regulatory requirements and the company's minimization of capital costs. To formally assess the relative performance of the considered models based on these loss functions, we apply the model confidence set (MCS) procedure by Hansen et al. (2011). As with the volatility predictions, this procedure allows us to determine SSM, the set of best models with a given confidence level.

The mean values of the loss functions FLF, C1, C2, and C3, outcomes of the MSC test, and ranking of the models belonging to SSM are given in Tables 21–24. These results point to several conclusions. First, the leading position in all rankings always belongs to one of the four proposed BM-RGARCH models. Second, several BM-RGARCH models usually belong to SSM. For the C1 function and ETH/USD cryptocurrency, all BM-RGARCH models belong to this set. Third, in the majority of cases, the representatives of the BM-RGARCH family are the only models belonging to SSM. In the case of the C1 function, there is no exception from this rule, i.e. none of the other models belongs to SSM for any of the cryptocurrencies. These conclusions demonstrate the practical advantage of the newly proposed approach over traditional models.

The above loss functions are relevant to regulatory requirements because they penalize the amount of loss connected with VaR exceedances. The severity of VaR exceedances is evaluated by various distance measures between realized returns and estimated VaR. This is in line with the current trend in the regulatory requirements, which is to move from measuring the number of exceedances towards measuring the amount of potential loss (Basel Committee on Banking Supervision, 2017). However, as of now, the basic system of backtesting rules imposed by the Basel Committee is still based on the number of VaR exceedances. Thus, according to these rules, we complete our analysis by performing the so-called Basel traffic light test. To this end, for each instrument, we compute the average capital charges, penalty, and the proportion of time spent in the green, yellow, and red zones. The penalty k and zone assignment result from the number of VaR exceedances in the last 250 days (Basel Committee on Banking Supervision, 1996), while the capital charges are computed according to the formula

$$cc^* = \min\left((3+k)cc, \widehat{VaR}_{0.01}^{t-1}(r_t)\right) \tag{27}$$

where k is the penalty, $cc = \frac{1}{60}\sum_{i=1}^{60}\widehat{VaR}_{0.01}^{t-i}(r_t)$ is the average 1% VaR over the last 60 days and $\widehat{VaR}_{0.01}^{t-1}(r_t)$ is the VaR forecast for the last day (see Liu and Luger, 2015 for the empirical application of this function).

The results of the procedure are presented in Table 25. The comparison of the proposed and the traditional models does not show dominance on any of the sides. For one of the instruments – ETC/USD – the proposed models are clearly superior – they give much lower capital charges (86–101 vs 124–151) with very safe penalty (resulting in around 70–80% of time in the green zone). For BTC/USD, the RGARCH models are slightly prevailing in terms of capital charges (49–57 vs 57–61), and for ETH/USD, the traditional models are slightly better than the proposed ones (60–81 vs 70–83). For the last currency – LTC/USD – the

Table 21
Evaluation of risk forecasts based on the firm's loss function FLF (Sarma et al., 2003).

Model	BTC/USD			ETC/USD			ETH/USD			LTC/USD		
	FLF	MCS p-value	Rank	FLF	MCS p-value	Rank	FLF	MCS p-value	Rank	FLF	MCS p-value	Rank
GARCH-n	13.067	0.000		19.857	0.000		15.553	0.001		16.224	0.100	
GARCH-t	13.811	0.000		24.637	0.000		16.735	0.000		16.668	0.000	
RGARCH-n	12.479	0.665*	2	19.741	0.000		15.881	0.000		16.442	0.000	
RGARCH-t	12.631	0.054		20.309	0.000		15.899	0.000		16.448	0.000	
BM-GARCH-n	13.034	0.000		24.421	0.000		15.224	0.011		16.587	0.000	
BM-GARCH-t	12.819	0.010		23.393	0.000		15.755	0.000		16.165	0.001	
BM-RGARCH-n	**12.432**	1.000*	1	18.261	0.082		14.896	0.197*	2	15.791	0.189*	2
BM-RGARCH-t	12.780	0.000		**17.693**	1.000*	1	16.738	0.000		19.810	0.000	
BMpc-RGARCH-n	12.652	0.133*	3	18.364	0.053		**14.790**	1.000*	1	**15.765**	0.173*	1
BMpc-RGARCH-t	12.796	0.000		17.888	0.082		16.327	0.000		15.586	1.000	

Notes: The table presents the values of the firm's loss function FLF for the considered cryptocurrencies (displayed at the top) corresponding to various GARCH models (displayed at the left). The lowest values of the loss function are marked in bold. Each value of the FLF is accompanied by the p-value from the MCS test. The MCS test is performed jointly for all models. The symbol * indicates that the model belongs to the set of best models with a confidence level of 0.90. The ranking is for models belonging to the set of best models.

Table 22
Evaluation of risk forecasts based on the firm's loss function C1 (Caporin, 2008).

Model	BTC/USD			ETC/USD			ETH/USD			LTC/USD		
	C1	MCS p-value	Rank	C1	MCS p-value	Rank	C1	MCS p-value	Rank	C1	MCS p-value	Rank
GARCH-n	0.786	0.000		0.801	0.000		0.749	0.000		0.752	0.006	
GARCH-t	0.797	0.000		0.839	0.000		0.772	0.000		0.761	0.000	
RGARCH-n	0.777	0.072		0.795	0.000		0.755	0.000		0.755	0.000	
RGARCH-t	0.779	0.000		0.801	0.000		0.758	0.000		0.756	0.000	
BM-GARCH-n	0.783	0.000		0.835	0.000		0.746	0.002		0.760	0.000	
BM-GARCH-t	0.779	0.000		0.828	0.000		0.756	0.000		0.752	0.000	
BM-RGARCH-n	**0.772**	1.000*	1	0.776	0.299*	3	**0.739**	1.000*	1	0.743	0.761*	2
BM-RGARCH-t	0.778	0.000		**0.774**	1.000*	1	0.766	0.000		0.795	0.000	
BMpc-RGARCH-n	0.777	0.000		0.778	0.173*	4	0.739	0.945*	2	0.743	0.761*	3
BMpc-RGARCH-t	0.777	0.001		0.775	0.476*	2	0.763	0.000		**0.742**	1.000*	1

Notes: The table presents the values of the firm's loss function C1, for the considered cryptocurrencies (displayed at the top) corresponding to various GARCH models (displayed at the left). The lowest values of the loss function are marked in bold. Each value of the C1 is accompanied by the p-value from the MCS test. The MCS test is performed jointly for all models. The symbol * indicates that the model belongs to the set of best models with a confidence level of 0.90. The ranking is for models belonging to the set of best models.

Table 23
Evaluation of risk forecasts based on the firm's loss function C2 (Caporin, 2008).

Model	BTC/USD			ETC/USD			ETH/USD			LTC/USD		
	C2	MCS p-value	Rank	C2	MCS p-value	Rank	C2	MCS p-value	Rank	C2	MCS p-value	Rank
GARCH-n	7.959	0.000		12.751	0.000		8.308	0.001		8.828	0.285*	4
GARCH-t	8.666	0.000		17.848	0.000		9.668	0.000		9.407	0.000	
RGARCH-n	7.371	0.219*	2	12.324	0.000		8.681	0.000		8.987	0.037	
RGARCH-t	7.469	0.000		13.010	0.000		8.849	0.000		9.087	0.000	
BM-GARCH-n	7.932	0.000		17.613	0.000		8.131	0.021		9.386	0.000	
BM-GARCH-t	7.714	0.000		16.615	0.000		8.814	0.000		9.054	0.002	
BM-RGARCH-n	**7.269**	1.000*	1	10.952	0.124*	2	7.849	0.676*	2	**8.634**	1.000*	1
BM-RGARCH-t	7.575	0.000		**10.766**	1.000*	1	9.793	0.000		12.901	0.000	
BMpc-RGARCH-n	7.497	0.001		11.154	0.009		**7.828**	1.000*	1	8.637	0.987*	3
BMpc-RGARCH-t	7.572	0.000		10.950	0.055		9.434	0.000		8.642	0.987*	2

Notes: The table presents the values of the firm's loss function C2, for the considered cryptocurrencies (displayed at the top) corresponding to various GARCH models (displayed at the left). The lowest values of the loss function are marked in bold. Each value of the C2 is accompanied by the p-value from the MCS test. The MCS test is performed jointly for all models. The symbol * indicates that the model belongs to the set of best models with a confidence level of 0.90. The ranking is for models belonging to the set of best models.

Table 24
Evaluation of risk forecasts based on the firm's loss function C3 (Caporin, 2008).

Model	BTC/USD			ETC/USD			ETH/USD			LTC/USD		
	C3	MCS p-value	Rank	C3	MCS p-value	Rank	C3	MCS p-value	Rank	C3	MCS p-value	Rank
GARCH-n	12.251	0.000		19.050	0.000		13.933	0.002		14.540	0.249*	4
GARCH-t	13.008	0.000		24.513	0.000		15.487	0.000		15.233	0.000	
RGARCH-n	11.598	0.023		18.564	0.000		14.360	0.000		14.740	0.050	
RGARCH-t	11.662	0.000		19.323	0.000		14.582	0.000		14.862	0.000	
BM-GARCH-n	12.178	0.000		24.245	0.000		13.753	0.024		15.216	0.000	
BM-GARCH-t	11.924	0.000		23.191	0.000		14.524	0.000		14.827	0.002	
BM-RGARCH-n	**11.412**	1.000*	1	17.021	0.103*	2	13.425	0.857*	2	14.337	0.569*	2
BM-RGARCH-t	11.749	0.000		**16.796**	1.000*	1	15.579	0.000		14.350	0.569*	3
BMpc-RGARCH-n	11.694	0.000		17.279	0.004		**13.415**	1.000*	1	**14.298**	1.000*	1
BMpc-RGARCH-t	11.750	0.000		17.017	0.042		15.202	0.000		19.035	0.000	

Notes: The table presents the values of the firm's loss function C3, for the considered cryptocurrencies (displayed at the top) corresponding to various GARCH models (displayed at the left). The lowest values of the loss function are marked in bold. Each value of the C3 is accompanied by the p-value from the MCS test. The MCS test is performed jointly for all models. The symbol * indicates that the model belongs to the set of best models with a confidence level of 0.90. The ranking is for models belonging to the set of best models.

proposed and traditional models exhibit similar results, especially when comparing the lowest attainable capital charges (69–104 and 67–82, respectively). The penalty stays safe for all cryptocurrencies in both compared groups of models, allowing them to stay around 60% of the time or more in a green zone.

5. Conclusions

Two characteristics that distinguish cryptocurrency returns from returns of traditional financial assets are huge volatility and large occasional price swings. Standard GARCH models do not capture the volatility dynamics of cryptocurrencies well, mostly because of the occurrence of extremely large observations (Charles and Darné, 2019). For this reason, we propose a new approach to model volatility, which combines the range-based volatility model with the robust estimation method. To make such an approach feasible, we modify the bounded M-estimator of Muler and Yohai (2008) and use it for the RGARCH model of Molnár (2016). Thanks to this merger, we use additional

Table 25
Evaluation of risk forecasts based on Basel backtesting rules.

Model	BTC/USD					ETC/USD				
	Capital charge	Penalty	Green	Yellow	Red	Capital charge	Penalty	Green	Yellow	Red
GARCH-n	59.66	0.03	0.92	0.08	0.00	124.39	0.06	0.86	0.14	0.00
GARCH-t	70.06	0.08	0.82	0.18	0.00	151.30	0.00	1.00	0.00	0.00
RGARCH-n	49.12	0.08	0.82	0.18	0.00	115.08	0.04	0.89	0.11	0.00
RGARCH-t	56.76	0.07	0.85	0.15	0.00	115.73	0.00	1.00	0.00	0.00
BM-GARCH-n	66.35	0.07	0.85	0.15	0.00	149.14	0.00	1.00	0.00	0.00
BM-GARCH-t	63.75	0.07	0.85	0.15	0.00	144.74	0.00	1.00	0.00	0.00
BM-RGARCH-n	59.13	0.11	0.80	0.20	0.00	101.56	0.11	0.77	0.23	0.00
BM-RGARCH-t	60.80	0.11	0.80	0.20	0.00	85.78	0.12	0.70	0.30	0.00
BMpc-RGARCH-n	58.19	0.11	0.80	0.20	0.00	97.96	0.18	0.67	0.33	0.00
BMpc-RGARCH-t	57.20	0.11	0.80	0.20	0.00	93.16	0.09	0.78	0.22	0.00
	ETH/USD					LTC/USD				
	Capital charge	Penalty	Green	Yellow	Red	Capital charge	Penalty	Green	Yellow	Red
GARCH-n	61.50	0.00	1.00	0.00	0.00	81.94	0.10	0.79	0.21	0.00
GARCH-t	73.70	0.00	1.00	0.00	0.00	79.37	0.05	0.87	0.13	0.00
RGARCH-n	81.22	0.00	0.99	0.01	0.00	73.34	0.01	0.99	0.01	0.00
RGARCH-t	65.60	0.00	1.00	0.00	0.00	67.44	0.00	1.00	0.00	0.00
BM-GARCH-n	60.47	0.01	0.99	0.01	0.00	69.99	0.01	0.99	0.01	0.00
BM-GARCH-t	79.38	0.12	0.73	0.27	0.00	80.25	0.09	0.77	0.23	0.00
BM-RGARCH-n	70.69	0.12	0.73	0.27	0.00	71.26	0.17	0.64	0.36	0.00
BM-RGARCH-t	82.82	0.08	0.83	0.17	0.00	103.96	0.00	1.00	0.00	0.00
BMpc-RGARCH-n	70.34	0.16	0.69	0.31	0.00	68.65	0.10	0.76	0.24	0.00
BMpc-RGARCH-t	79.03	0.14	0.72	0.28	0.00	76.07	0.20	0.58	0.42	0.00

Notes: The table presents an evaluation of the risk forecasts produced by the competing models (displayed at the left) for the considered instruments BTC/USD, ETC/USD, ETH/USD, and LTC/USD (displayed at the top of each panel), according to the Basel backtesting rules. For each instrument and model, it gives the average capital charge and penalty incurred over the last 250 trading days, as well as the proportion of time spent in green, yellow, and red penalty zones. All quantities are calculated for the whole forecasting period, shortened by the window of 250 initial observations needed to compute the first penalty.

information commonly available alongside daily closing prices, i.e., daily low and high prices, and at the same time, we limit the influence of extreme observations on the estimation results. This procedure is not as sensitive to outliers as the maximum likelihood estimation of the range-based models. We also propose to introduce an additional feature to the bounded M-estimator, which allows us to account for the size of the outliers. In this proposition, we suggest not limiting the outliers to the specified value but decreasing them proportionally to their size. It reflects the observations of financial markets, where, after extreme events, the volatility persists at an increased level.

We apply the suggested range-based volatility model with the modified robust estimation method to four cryptocurrencies: Bitcoin, Ethereum Classic, Ethereum, and Litecoin. We compare the proposed approach with other approaches that differ in the model specification, distribution of the error term, and estimation method. The results show huge differences among estimated values of parameters for the applied models. Such differences expose the practical importance of the model choice and a relevant estimation method.

We evaluate the accuracy of forecasts from all considered volatility models. We find that forecasts of volatility and risk based on the proposed approach are more accurate than forecasts from the three applied benchmarks: the standard GARCH model, the standard range-based GARCH model, and the GARCH model with robust estimation. The best of all analyzed models is the RGARCH model with the normal distribution of error term and the corrected BM estimator with the proportional cut.

Our approach can be extended in the future to model other properties of financial time series like leverage effect or long memory. Modeling such features can further improve the forecasting accuracy of the suggested approach. The proposed method can also be applied to other assets like stocks, traditional currencies, or commodities in order to investigate whether it is equally as successful as it is for cryptocurrencies. Another potential line of future research is an integration of the range-based GARCH models with alternative robust methods such as those based on the testing approach (Franses and Ghijsels, 1999) or wavelet outlier detection with soft or hard thresholding (Grané and Veiga, 2014). Our proposition can also be extended to a multivariate approach, where the range-based models can be combined with robust DCC/CDCC estimation (Boudt et al., 2013) or the methods dedicated to high-dimensional volatility problems, such as the robust principal volatility component analysis (Trucíos et al., 2019). The economic value of our suggested approach can also be assessed based on the utility measure of West et al. (1993) (see You and Liu, 2020).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The link to data is given in the paper in Section 3.1.

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Appendix A

Here, we present the results of using a fully parametric and fully nonparametric approach to evaluate VaR and ES. In the parametric approach, we combine the volatility forecasts obtained from the ten models considered in the paper with the parametric estimation of the distribution quantile and conditional expectation. The estimation of quantile and conditional expectation involves two types of distribution, normal and Student's t, which correspond to the distributional assumptions used to estimate the variance. The initial evaluation of VaR and ES forecasts is based on a statistical testing procedure. First, we conduct VaR tests of unconditional coverage and independence, then ES tests, and finally, we use the encompassing VaR and ES tests.

The results of testing VaR and ES separately show that the parametric approach fails in many cases. According to the Kupiec test, VaR forecasts are unsatisfactory in cases of all robust models, both the proposed and traditional ones, and among the non-robust models, they are unsatisfactory for models with the normal distribution (Table A1). ES forecasts fail for all models with the normal distribution (Table A2).

With regard to the encompassing test (Table A3), the dominating result for joint VaR and ES forecasts is that the proposed BM-RGARCH models encompass their competitors. Most often, they encompass all kinds of competitors, the basic GARCH models, the RGARCH models and the robust BM-GARCH models, regardless of the distributional assumption. Sometimes, the combination of the proposed model and one of the competitors turns out superior. However, the situations in which the proposed models are encompassed are rather exceptional. Similarly, for the auxiliary ES variant of the test, the proposed models usually encompass all competitors, and apart from inconclusive cases, other results are scarce. This suggests that the information content included in the proposed models is crucial for forecasting VaR and ES.

Summarizing the results for the parametric VaR and ES forecasts, none of the models are satisfactory. On the one hand, the proposed approach, linking RGARCH models with the robust BM estimation, is superior compared to the traditional one (as shown by the encompassing test), but, on the other hand, the forecasts coming from this approach cannot be accepted in the light of the basic statistical tests. Thus, we have decided to employ nonparametric and semiparametric methods to forecast VaR and ES.

The results of the statistical tests for VaR and ES forecasts coming from the fully nonparametric HS method (Tables A1 and A2) are still not satisfactory. As shown by the Kupiec test, HS forecasts fail in terms of the number of VaR exceedances. According to the other VaR tests, the independence of VaR exceedances is rejected in two cases out of four. The ES tests are passed only for one instrument.

The outcomes connected with the semiparametric approach are presented in Section 4. Here, for completeness of the statistical testing procedure, we present the results of the encompassing test that checks the proposed models estimated by the semiparametric FHS method against the HS models.

The results of the encompassing test comparing the proposed FHS models vs the HS benchmark (Table A4) are similar to those from testing the proposed parametric models against their parametric competitors. Namely, either the proposed models are encompassing, or the combination of the models prevails. Thus, this test confirms that the information content of the proposed models is essential to predict risk accurately.

Table A1
Evaluation of parametric VaR forecasts based on the Kupiec (1995). generalized Markov (Christoffersen, 1998) and corrected geometric (Malecka, 2022) tests.

Model	BTC/USD			ETC/USD			ETH/USD			LTC/USD		
	Kupiec	Gen. Markov	Geometric	Kupiec	Gen. Markov	Geometric	Kupiec	Gen. Markov	Geometric	Kupiec	Gen. Markov	Geometric
HS	0.058	0.020	0.007	0.002	0.206	0.369	0.000	0.011	0.031	0.002	0.571	0.320
GARCH-n	0.019	0.998	0.498	0.045	0.597	0.422	0.021	0.766	0.499	0.103	0.825	0.498
GARCH-t	0.154	0.609	0.500	0.551	0.733	0.402	0.390	0.290	0.500	0.617	0.713	0.394
RGARCH-n	0.019	0.998	0.499	0.239	0.971	0.406	0.127	0.446	0.500	0.174	0.938	0.500
RGARCH-t	0.488	0.358	0.498	0.732	0.403	0.213	0.597	0.379	0.500	0.910	0.491	0.318
BM-GARCH-n	0.000	0.308	0.499	0.000	0.199	0.499	0.000	0.248	0.456	0.000	0.615	0.260
BM-GARCH-t	0.000	0.599	0.499	0.000	0.075	0.500	0.005	0.986	0.498	0.000	0.546	0.500
BM-RGARCH-n	0.000	0.560	0.499	0.000	0.051	0.500	0.000	0.501	0.499	0.000	0.848	0.405
BM-RGARCH-t	0.000	0.260	0.499	0.000	0.258	0.499	0.002	0.539	0.499	0.001	0.750	0.499
BMpc-RGARCH-n	0.000	0.297	0.499	0.000	0.268	0.500	0.000	0.215	0.356	0.000	0.461	0.499
BMpc-RGARCH-t	0.000	0.622	0.499	0.000	0.258	0.499	0.000	0.106	0.333	0.001	0.750	0.499

Notes: The table presents the p-values of the three VaR tests (displayed at the top) for the considered cryptocurrencies (displayed at the top), corresponding to various GARCH models (displayed at the left).

Table A2
Evaluation of parametric ES forecasts based on the residual-based test U (McNeil and Frey, 2000) and ES-prediction-corrected variables test U^{ES} (Małecka, 2020).

Model	BTC/USD		ETC/USD		ETH/USD		LTC/USD	
	U	U^{ES}	U	U^{ES}	U	U^{ES}	U	U^{ES}
HS	0.085	0.057	0.036	0.016	0.293	0.081	0.354	0.139
GARCH-n	0.001	0.001	0.000	0.000	0.034	0.032	0.002	0.001
GARCH-t	0.940	0.924	1.000	1.000	0.928	0.898	0.961	0.901
RGARCH-n	0.001	0.001	0.000	0.001	0.011	0.011	0.003	0.004
RGARCH-t	0.834	0.829	1.000	0.995	0.966	0.957	0.886	0.863
BM-GARCH-n	0.000	0.000	0.000	0.000	0.001	0.001	0.000	0.000
BM-GARCH-t	0.403	0.428	0.736	0.750	0.675	0.690	0.779	0.779
BM-RGARCH-n	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
BM-RGARCH-t	0.489	0.506	0.794	0.796	0.594	0.590	0.763	0.768
BMpc-RGARCH-n	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
BMpc-RGARCH-t	0.348	0.359	0.835	0.829	0.707	0.708	0.793	0.805

Notes: The table presents the p-values of the two ES tests (displayed at the top) for the considered cryptocurrencies (displayed at the top), corresponding to various GARCH models (displayed at the left).

Table A3
Evaluation of the proposed parametric VaR and ES models vs the standard parametric models based on the encompassing test (Dimitriadis et al., 2023).

Proposed model	Joint VaR and ES test				Auxiliary ES test				Avg. weights
	E'ing	E'd	Comb	Incon	E'ing	E'd	Comb	Incon	
Alternative model GARCH-n									
BM-RGARCH-n	2	0	2	0	3	0	1	0	(0.806, 0.000)
BMpc-RGARCH-n	2	0	2	0	3	0	1	0	(0.797, 0.000)
Alternative model GARCH-t									
BM-RGARCH-t	2	0	2	0	4	0	0	0	(0.977, 0.880)
BMpc-RGARCH-t	2	0	2	0	4	0	0	0	(0.985, 0.879)
Alternative model RGARCH-n									
BM-RGARCH-n	2	0	2	0	3	0	1	0	(0.774, 0.000)
BMpc-RGARCH-n	4	0	0	0	4	0	0	0	(0.918, 0.000)
Alternative model RGARCH-t									
BM-RGARCH-t	1	0	2	1	3	0	0	1	(0.968, 0.829)
BMpc-RGARCH-t	2	0	2	0	4	0	0	0	(0.972, 0.788)
Alternative model BM-GARCH-n									
BM-RGARCH-n	3	0	1	0	0	0	0	4	(0.868, 0.000)
BMpc-RGARCH-n	3	1	0	0	0	0	0	4	(0.795, 0.000)
Alternative model BM-GARCH-t									
BM-RGARCH-t	0	1	3	0	0	0	0	4	(0.567, 0.019)
BMpc-RGARCH-t	2	1	1	0	0	0	0	4	(0.645, 0.236)

Notes: The table summarizes testing the proposed parametric models (displayed at the left) against the standard parametric models (displayed at the top of the panels). Test results of the joint VaR and ES and the auxiliary ES encompassing tests for all considered instruments, BTC/USD, ETC/USD, ETH/USD, and LTC/USD, are based on the no-crossing link function. The "E'ing" column gives the number of occurrences (out of four instruments where the proposed model encompasses a competing model, the "E'd" column counts the occurrences where the proposed model is encompassed, "Comb" - neither model encompasses its competitor, and "Incon" - both models encompass each other. The column "Avg. weights" shows the estimated combination weights of the no-crossing link function ($\theta_1 .\theta_2$) averaged over the four instruments.

Table A4
Evaluation of the proposed semiparametric VaR and ES models vs the historical simulation model based on the encompassing test (Dimitriadis et al., 2023).

Proposed model	Joint VaR and ES test				Auxiliary ES test				Avg. weights
	E'ing	E'd	Comb	Incon	E'ing	E'd	Comb	Incon	
BM-RGARCH-n	1	0	3	0	3	0	1	0	(0.922, 0.957)
BM-RGARCH-t	0	0	4	0	1	0	3	0	(0.89, 0.926)
BMpc-RGARCH-n	3	0	1	0	3	0	1	0	(0.91, 0.952)
BMpc-RGARCH-n	3	0	1	0	2	0	2	0	(0.905, 0.934)

Notes: The table summarizes testing the proposed models estimated by the semiparametric FHS method (displayed at the left) against the nonparametric HS model. Test results of the joint VaR and ES and the auxiliary ES encompassing tests for all considered instruments, BTC/USD, ETC/USD, ETH/USD, and LTC/USD, are based on the no-crossing link function. The "E'ing" column gives the number of occurrences (out of four instruments where the proposed model encompasses a competing model. The "E'd" column counts the occurrences where the proposed model is encompassed, "Comb" - neither model encompasses its competitor, and "Incon" - both models encompass each other. The column "Avg. weights" shows the estimated combination weights of the no-crossing link function ($\theta_1 .\theta_2$) averaged over the four instruments.

References

Alexander, C., Dakos, M., 2020. A critical investigation of cryptocurrency data and analysis. Quant. Finance 20 (2), 173–188.
Almeida, J., Gonçalves, T.C., 2022. A systematic literature review of volatility and risk management on cryptocurrency investment: a methodological point of view. Risks 10 (5), 107.
Bariviera, A.F., Merediz-Solà, I., 2021. Where do we stand in cryptocurrencies economic research? A survey based on hybrid analysis. J. Econ. Surv. 35 (2), 377–407.
Basel Committee on Banking Supervision, 1996. Supervisory framework for the use of 'backtesting' in conjunction with the internal models approach to market risk capital requirements. Retrieved from. https://www.bis.org/publ/bcbs22.htm.
Basel Committee on Banking Supervision, 2017. High-level Summary of Basel III Reforms. Online, Basel, Switzerland. Retrieved from. https://www.bis.org/bcbs/publ/d424_hlsummary.pdf.
Bei, S., Yang, A., Pei, H., Si, X., 2023. Price risk analysis using GARCH family models: evidence from Shanghai crude oil futures market. Econ. Modell. 125, 106367.
Berkowitz, J., Christoffersen, P., Pelletier, D., 2011. Evaluating value-at-risk models with desk-level data. Manag. Sci. 57 (12), 2213–2227.
Bollerslev, T., 1986. Generalised autoregressive conditional heteroscedasticity. J. Econom. 31, 307–327.
Boudt, K., Croux, C., Laurent, S., 2011. Outlyingness weighted covariation. J. Financ. Econom. 9 (4), 657–684.
Boudt, K., Daníelsson, J., Laurent, S., 2013. Robust forecasting of dynamic conditional correlation GARCH models. Int. J. Forecast. 29 (2), 244–257.
Bouri, E., Shahzad, S.J.H., Roubaud, D., 2019. Co-explosivity in the cryptocurrency market. Finance Res. Lett. 29, 178–183.
Brini, A., Lenz, J., 2024. Pricing cryptocurrency options with machine learning regression for handling market volatility. Econ. Modell. 136, 106752.
Caporin, M., 2008. Evaluating value-at-risk measures in the presence of long memory conditional volatility. J. Risk 10 (3), 79–110.
Carnero, M.A., Peña, D., Ruiz, E., 2007. Effects of outliers on the identification and estimation of GARCH models. J. Time Anal. 28 (4), 471–497.
Carnero, M.A., Peña, D., Ruiz, E., 2012. Estimating GARCH volatility in the presence of outliers. Econ. Lett. 114 (1), 86–90.
Catalán, B., Trívez, F.J., 2007. Forecasting volatility in GARCH models with additive outliers. Quant. Finance 7 (6), 591–596.
Catania, L., Grassi, S., 2022. Forecasting cryptocurrency volatility. Int. J. Forecast. 38 (3), 878–894.
Catania, L., Grassi, S., Ravazzolo, F., 2018. Predicting the volatility of cryptocurrency time-series. In: Corazza, M., Durbán, M., Grané, A., Perna, C., Sibillo, M. (Eds.), Mathematical and Statistical Methods for Actuarial Sciences and Finance. Springer International Publishing, pp. 203–207.
Chaim, P., Laurini, M.P., 2018. Volatility and return jumps in Bitcoin. Econ. Lett. 173, 158–163.
Charles, A., Darné, O., 2019. Volatility estimation for Bitcoin: replication and robustness. International Economics 157, 23–32.
Chou, R.Y., 2005. Forecasting financial volatilities with extreme values: the conditional autoregressive range (CARR) model. J. Money Credit Bank. 37 (3), 561–582.
Corbet, S., Lucey, B., Urquhart, A., Yarovaya, L., 2019. Cryptocurrencies as a financial asset: a systematic analysis. Int. Rev. Financ. Anal. 62, 182–199.
Dimitriadis, T., Liu, X., Schnaitmann, J., 2023. Encompassing tests for value at risk and expected shortfall multistep forecasts based on inference on the boundary. J. Financ. Econom. 21 (2), 412–444.

Dudek, G., Fiszeder, P., Kobus, P., Orzeszko, W., 2024. Forecasting cryptocurrencies volatility using statistical and machine learning methods: a comparative study. Appl. Soft Comput. 151, 111132.

Fałdziński, M., Fiszeder, P., Molnár, P., 2024. Improving volatility forecasts: evidence from range-based models. N. Am. J. Econ. Finance 69, 102019.

Fang, F., Ventre, C., Basios, M., Kanthan, L., Martinez-Rego, D., Wu, F., Li, L., 2022. Cryptocurrency trading: a comprehensive survey. Financial Innovation 8, 13.

Fiszeder, P., Fałdziński, M., Molnár, P., 2019. Range-based DCC models for covariance and value-at-risk forecasting. J. Empir. Finance 54, 58–76.

Fiszeder, P., Małecka, M., 2022. Forecasting volatility during the outbreak of Russian invasion of Ukraine: application to commodities, stock indices, currencies, and cryptocurrencies. Equilibrium. Quarterly Journal of Economics and Economic Policy 17 (4), 939–967.

Fiszeder, P., Perczak, G., 2016. Low and high prices can improve volatility forecasts during the turmoil period. Int. J. Forecast. 32 (2), 398–410.

Floros, C., Gkillas, K., Konstantatos, C., Tsagkanos, A., 2020. Realized measures to explain volatility changes over time. J. Risk Financ. Manag. 13 (6), 125.

Franses, P.H., Ghijsels, H., 1999. Additive outliers, GARCH and forecasting volatility. Int. J. Forecast. 15 (1), 1–9.

Francq, C., Zakoïan, J.-M., 2020. Virtual historical simulation for estimating the conditional VaR of large portfolios. J. Econom. 217 (2), 356–380.

Garman, M.B., Klass, M.J., 1980. On the estimation of security price volatilities from historical data. J. Bus. 53 (1), 67–78.

Ghorbel, A., Trabelsi, A., 2014. Energy portfolio risk management using time-varying extreme value copula methods. Econ. Modell. 38, 470–485.

Giacomini, R., White, H., 2006. Tests of conditional predictive ability. Econometrica 74 (6), 1545–1578.

Grané, A., Veiga, H., 2014. Outliers, GARCH-type models and risk measures: a comparison of several approaches. J. Empir. Finance 26, 26–40.

Hall, M.K., Jasiak, J., 2024. Modelling common bubbles in cryptocurrency prices. Econ. Modell. 139, 106782.

Hansen, P.R., Lunde, A., Nason, J.M., 2011. The model confidence set. Econometrica 79, 453–497.

Hung, J.-C., Liu, H.-C., Yang, J.J., 2020. Improving the realized GARCH's volatility forecast for Bitcoin with jump-robust estimators. N. Am. J. Econ. Finance 52, 101165.

Kambouroudis, D.S., McMillan, D.G., Tsakou, K., 2021. Forecasting realized volatility: the role of implied volatility, leverage effect, overnight returns, and volatility of realized volatility. J. Futures Mark. 41, 1618–1639.

Kayal, P., Rohilla, P., 2021. Bitcoin in the economics and finance literature: a survey. SN Business & Economics 1, 88.

Ke, R., Jia, J., Tan, C., 2021. Robust minimum distance estimators for the CARR(1,1) model. J. Stat. Comput. Simulat. 91 (3), 564–580.

Kupiec, P.H., 1995. Techniques for verifying the accuracy of risk measurement models. J. Deriv. 3 (2), 73–84.

Kyriazis, N.A., 2019. A survey on empirical findings about spillovers in cryptocurrency markets. J. Risk Financ. Manag. 12 (4), 170.

Liu, X., Luger, R., 2015. Unfolded GARCH models. J. Econ. Dynam. Control 58, 186–217.

Louzis, D.P., Xanthopoulos-Sisinis, S., Refenes, A.P., 2014. Realized volatility models and alternative Value-at-Risk prediction strategies. Econ. Modell. 40, 101–116.

Małecka, M., 2020. Extremal risk management: expected shortfall value verification using the bootstrap method. J. Comput. Finance 23 (4), 35–39.

Malecka, M., 2022. Asymptotic properties of duration-based VaR backtests. Stat. Risk Model. 39 (3–4), 49–73.

McNeil, A.J., Frey, R., 2000. Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. J. Empir. Finance 7 (3–4), 271–300.

Merediz-Solà, I., Bariviera, A.F., 2019. A bibliometric analysis of Bitcoin scientific production. Res. Int. Bus. Finance 50, 294–305.

Molnár, P., 2012. Properties of range-based volatility estimators. Int. Rev. Financ. Anal. 23, 20–29.

Molnár, P., 2016. High-low range in GARCH models of stock return volatility. Appl. Econ. 48 (51), 4977–4991.

Muler, N., Yohai, V.J., 2008. Robust estimates for GARCH models. J. Stat. Plann. Inference 138 (10), 2918–2940.

Müller, F.M., Santos, S.S., Righi, M.B., 2023. A description of the COVID-19 outbreak role in financial risk forecasting. N. Am. J. Econ. Finance 66, 101894.

Nelson, D.B., Cao, C.Q., 1992. Inequality constraints in the univariate GARCH model. J. Bus. Econ. Stat. 10, 229–235.

Nolde, N., Ziegel, J.F., 2017. Elicitability and backtesting: perspectives for banking regulation. Ann. Appl. Stat. 11 (4), 1833–1874.

Pajhede, T., 2017. Backtesting value-at-risk: a generalized Markov test. J. Forecast. 36 (5), 597–613.

Parkinson, M., 1980. The extreme value method for estimating the variance of the rate of return. J. Bus. 53 (1), 61–65.

Patton, A.J., Ziegel, J.F., Chen, R., 2019. Dynamic semiparametric models for expected shortfall (and Value-at-Risk). J. Econom. 211 (2), 388–413.

Reschenhofer, E., Mangat, M.K., Stark, T., 2020. Volatility forecasts, proxies and loss functions. J. Empir. Finance 59, 133–153.

Rossi, B., Sekhposyan, T., 2011. Understanding models' forecasting performance. J. Econom. 164 (1), 158–172.

Sarma, M., Thomas, S., Shah, A., 2003. Selection of value-at-risk models. J. Forecast. 22 (4), 337–358.

Shen, Y., 2018. International risk transmission of stock market movements. Econ. Modell. 69, 220–236.

Tan, S.K., Chan, J.S.K., Ng, K.H., 2020. On the speculative nature of cryptocurrencies: a study on Garman and Klass volatility measures. Finance Res. Lett. 32, 101075.

Taylor, J.W., 2019. Forecasting value at risk and expected shortfall using a semiparametric approach based on the asymmetric Laplace distribution. J. Bus. Econ. Stat. 37 (1), 121–133.

Trucíos, C., 2019. Forecasting Bitcoin risk measures: a robust approach. Int. J. Forecast. 35 (3), 836–847.

Trucíos, C., Hotta, L.K., 2015. Bootstrap prediction in univariate volatility models with leverage effect. Math. Comput. Simulat. 120, 91–103.

Trucíos, C., Hotta, L.K., Pereira, P.L.V., 2019. On the robustness of the principal volatility components. J. Empir. Finance 52, 201–219.

West, K.D., Edison, H.J., Cho, D., 1993. A utility-based comparison of some models of exchange rate volatility. J. Int. Econ. 35 (1–2), 23–45.

Wu, X., Wu, M., 2022. Forecasting renminbi exchange rate volatility using CARR-MIDAS model. Complexity, 3127761.

Wu, X., Niu, S., Xie, H., 2020. Forecasting Bitcoin volatility using two-component. Econ. Comput. Econ. Cybern. Stud. Res. 54 (3), 77–94.

Xie, H., 2019. Financial volatility modeling: the feedback asymmetric conditional autoregressive range model. J. Forecast. 38, 11–28.

You, Y., Liu, X., 2020. Forecasting short-run exchange rate volatility with monetary fundamentals: a GARCH-MIDAS approach. J. Bank. Finance 116, 105849.

Zhang, Y., Ma, F., Liao, Y., 2020. Forecasting global equity market volatilities. Int. J. Forecast. 36 (4), 1454–1475.

Zhou, J., 2014. Modeling conditional covariance for mixed-asset portfolios. Econ. Modell. 40, 242–249.